Concave Envelope Analysis in Nonconvex Optimisation

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Abstract

Common solution techniques for minimum cost nonconvex optimisation problems, such as branch and bound, cone covering, outer approximation, and inner approximation methodologies, involve the use and solution of relaxations of the original problem. These relaxations generally involve the construction of the convex envelopes or hulls of (some part of) the original problem's objective function, feasible region, or both. One disadvantage of using convex envelopes or hulls is that the shape or behaviour of the objective function or feasible region between extreme points is not explicitly taken into account. In this paper, we develop the concept of underestimating "concave envelopes". Similar in concept to convex envelopes, concave envelopes incorporate non-extreme point information in their construction. We use concave envelopes to develop a method of parametric analysis for a class of nonconvex optimisation problems.

1 Introduction

In this paper we consider deterministic minimum cost optimisation problems in which the cost functions and/or feasible regions are nonconvex. Such optimisation problems arise in a variety of contexts involving discounting or other economies of scale. Example application areas include waste-disposal and management systems [4, 10], natural gas pipeline systems [10], product distribution and transportation [5, 11], manufacturing [14], telecommunications [7, 15], and industry regulation analysis [12].

Many of the solution approaches discussed in the literature (see, for example, [3, 9]) for nonconvex optimisation problems involve the use and solution of "relaxations" of (some part of) the original problem. These relaxations generally involve either a simpler objective function or a simpler (and larger) feasible region than in the original problem. For example, outer approximation algorithms involve the construction of larger feasible regions that contain the part of the feasible region of

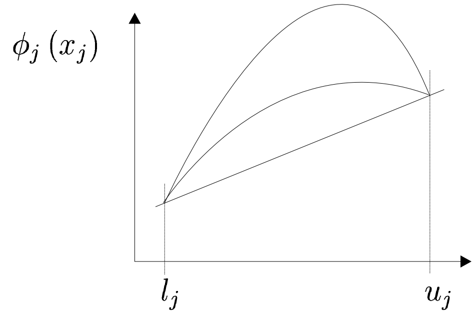

Figure 1: Two different concave functions with same convex envelope

the original problem being searched. The relaxed feasible region is constructed so that minimisation of the objective function over this new feasible set is comparatively simple. The branch and bound solution methodology solves a relaxation of the original problem at each branch and bound node in which the objective function of the original problem is replaced with its convex envelope. The solution to this relaxation provides a lower bound to the solution of the subproblem at that node of the branch and bound tree.

Such relaxations generally involve the construction of the convex envelopes or hulls of (some part of) the original problem's objective function, feasible region, or both. One disadvantage of using convex envelopes or hulls is that the shape or behaviour of the objective function or feasible region between extreme points is not explicitly taken into account. For example, consider that the two concave functions illustrated in Figure 1 have identical convex envelopes.

In this paper we develop the concept of underestimating "concave envelopes" for a specific class of nonconvex optimisation problems. Similar in concept to convex envelopes, underestimating concave envelopes take account of the shape of the concave objective function over the problem's feasible region. First, we formulate the class of problem considered in this paper. We then present the convex envelope relaxation used in common solution algorithms for problems of this type. Next, the concept of underestimating concave envelopes is defined, and a new relaxation based on concave envelopes is developed. This relaxation is used to develop a powerful method of parametric analysis for the problems considered in this paper. Finally, applications of concave envelopes to solution algorithms for this class of problems are briefly discussed.

2 Problem Formulation

In this paper we are concerned with problems of the following form:

$$(Q) \qquad \min \phi\left(\underline{x}\right) \text{ s.t. } \left(\underline{x}\right) \in S = X \cap H, \underline{x} = \left(\underline{x}_{\mathbb{R}}, \underline{x}_{\mathbb{Z}}\right) \in \mathbb{Y}^{n+m}$$

where $\underline{x} = \left(\underline{x}_{\mathbb{R}}, \underline{x}_{\mathbb{Z}}\right) = \left(\ldots, x_j, \ldots\right)^{\mathrm{T}}$ is a vector of decision variables with index set $J = \{1, \ldots, j, \ldots, n+m\}$, $\mathbb{Y}^{n+m}$ is the subspace of $\mathbb{R}^{n+m}$ such that $\mathbb{Y}^{n+m} = \{\mathbb{R}^n : \mathbb{Z}^m\}$, $\underline{x}_{\mathbb{R}} \in \mathbb{R}^n$ is the vector of continuous valued solution variables, $\underline{x}_{\mathbb{Z}} \in \mathbb{Z}^m$

is the vector of integer valued solution variables, $\phi(\underline{x})$ is a separable concave real-valued function defined on the decision variables $\underline{x}$ that performs the mapping $\phi : \mathbb{R}^{n+m} \to \mathbb{R}$, and $S = X \cap H$ defines the feasible region of problem Q. X is the set defined as:

$$X = \left\{\underline{x} = (\underline{x}_{\mathbb{R}}, \underline{x}_{\mathbb{Z}}) : \left\{\underline{\gamma}(\underline{x}) \ = \underline{g}\right\}\right\}$$

where $\underline{\gamma}(\underline{x})$ is the vector of real valued (possibly nonlinear) functions of $\underline{x}$ and $\underline{g}$ is a vector of real valued scalars that together define the constraints of problem Q. H is the hyperrectangle defined by the simple flow bounds on $\underline{x}$ as follows

$$H = \{\underline{x} : \underline{l} \leq \underline{x} \leq \underline{u}\}$$

where $\underline{l} = (\ldots, l_j, \ldots)^{\mathrm{T}} \in \mathbb{R}^{n+m}$ is the lower bound vector for the decision variable vector $\underline{x}$, and $\underline{u} = (\ldots, u_j, \ldots)^{\mathrm{T}} \in \mathbb{R}^{n+m}$ is the upper bound vector for the decision variable vector $\underline{x}$.

In addition, throughout this paper, for any problem $\bullet$ let $\nu[\bullet]$ denote the optimal objective function value of $\bullet$. Also, for any problem $\bullet$, let $lb[\bullet]$ denote a lower bound to $\nu[\bullet]$, and let $ub[\bullet]$ denote an upper bound to $\nu[\bullet]$.

3 Convex Envelopes

We begin with the following formal definition of a convex envelope:

Definition 3.1 (see [8]) *Let $f : S \to \mathbb{R}$ be a lower semi-continuous function, where S is a nonempty convex set in $\mathbb{R}^n$. The* convex envelope *of $f(\underline{x})$ taken over S is a function $\bar{f}(\underline{x})$ such that*

(i) $\bar{f}(\underline{x})$ is convex on S.

(ii) $\bar{f}(\underline{x}) \leq f(\underline{x}) \ \forall \ \underline{x} \in S$.

(iii) If $h(\underline{x})$ is any convex function defined on S such that $h(\underline{x}) \leq f(\underline{x}) \forall \underline{x} \in S$, then $h(\underline{x}) \leq \bar{f}(\underline{x}) \forall \underline{x} \in S$.

We can now specify a relaxation of Q that uses convex envelopes. We first define the set $\bar{X} \subset \mathbb{R}^{n+m}$ as follows:

$$\bar{X} = \left\{(\underline{x}) : \left\{C\underline{x} \ = \underline{\bar{g}}\right\}\right. \tag{1}$$

where C is a matrix of elements $c_{ij} \in \mathbb{R}$, and $\underline{\bar{g}}$ is a vector of real valued scalars. $C\underline{x} = \underline{\bar{g}}$ is defined such that $X \subseteq \bar{X}$. We then define $\bar{S} \in \mathbb{R}^{n+m}$ such that $\bar{S} = \bar{X} \cap H$. Note that $S \subseteq \bar{S}$.

The convex envelope relaxation of Q presented here, denoted $\bar{Q}$, is defined as:

$$(\bar{Q}) \qquad \min \bar{\phi}(\underline{x}) \ \text{ s.t. } \ \underline{x} \in \bar{S} = \bar{X} \cap H, \ \underline{x} \in \mathbb{R}^{n+m}$$

where $\bar{\phi}(\underline{x})$ is the convex envelope of $\phi(\underline{x})$ on H. Because of the separability of $\phi(\underline{x})$, we have $\bar{\phi}(\underline{x}) = \sum_{j \in J} \bar{\phi}_j(x_j)$ where $\bar{\phi}_j(x_j)$ is the convex envelope of $\phi_j(x_j)$ over the range $l_j \leq x_j \leq u_j$. Note that because the objective functions considered here are either separable linear or separable concave functions, their convex envelopes are in fact affine (linear) functions. That is

$$\bar{\phi}_j(x_j) = f_j + c_j \cdot x_j \tag{2}$$

where

$$\begin{aligned} c_j &= \frac{\phi_j(u_j) - \phi_j(l_j)}{u_j - l_j} \\ f_j &= \phi_j(l_j) - c_j \cdot l_j \end{aligned} \tag{3}$$

Since each $\bar{\phi}_j(x_j)$ is affine and $\bar{S}$ is a polytope in $\mathbb{R}^{n+m}$, problem $\bar{Q}$ is a linear program and is easily solved. Hence, problem $\bar{Q}$ is referred to as the *linear relaxation* of problem Q.

Let $\bar{\underline{x}} = (\ldots, \bar{x}_j, \ldots) \in \mathbb{R}^{n+m}$ be the optimal solution vector for problem $\bar{Q}$, let a_{ij} denote the element in the i-th row and j-th column in the simplex tableau of problem $\bar{Q}$, let $\bar{a}_{ij}$ denote the element in the i-th row and j-th column in the *optimal* simplex tableau of problem $\bar{Q}$, and let $\bar{\pi}_i$ denote the dual variable associated with the i-th row of the optimal simplex tableau of problem $\bar{Q}$. The reduced cost associated with variable x_j in the optimal solution to problem $\bar{Q}$, denoted $\bar{r}_j$, is given by $\bar{r}_j = c_j - \bar{z}_j$, where $\bar{z}_j = \sum_{i \in I} \bar{\pi}_i \cdot a_{ij}$. In addition, we partition the index set J into $J = B \cup NL \cup NU$ where $j \in B$ if x_j is a basic variable, $j \in NL$ if x_j is a nonbasic variable at its lower bound l_j, and $j \in NU$ if x_j is a nonbasic variable at its upper bound u_j in the optimal solution to problem $\bar{Q}$. The optimal solution to problem $\bar{Q}$ is used in defining the nonconvex relaxation of Q discussed in the following Section.

4 Underestimating Concave Envelope

In this Section, we define the concept of underestimating concave envelopes, and develop a nonlinear relaxation of problem Q that utilises underestimating concave envelopes. The definition of the underestimating concave envelope of a separable concave function is problem specific. In general, the specification of the underestimating concave envelope $\hat{\phi}_j(x_j)$ of the concave objective function $\phi_j(x_j)$ associated with variable j is predicated on the optimal solution to the linear relaxation $\bar{Q}$. To define $\hat{\phi}_j(x_j)$, it is useful to define another function, referred to as the *reduced cost function*, associated with $\hat{\phi}_j(x_j)$. The reduced cost function for any $j \in J$ is denoted $\Delta_j(x_j)$ and is defined as

$$\Delta_j(x_j) = \hat{\phi}_j(x_j) - \bar{z}_j \cdot x_j \tag{4}$$

The reduced cost *function* is a straight forward extension of the reduced cost *coefficient* used in linear programming. In fact, if $\hat{\phi}_j(x_j)$ is affine, then $\Delta_j(x_j) = \bar{r}_j \cdot x_j$

where $\bar{r}_j$ is the reduced cost coefficient associated with variable j in the optimal solution to problem $\bar{Q}$.

The underestimating concave envelope of $\phi_j(x_j)$ can then be defined as follows.

Definition 4.1 *Let Q be a problem of the type defined in Section 2 and $\bar{Q}$ be the linear relaxation of problem Q as defined in Section 3. Let $\phi_j : H_j \to \mathbb{R}$ be the lower semi-continuous concave objective function defined on variable j in problem Q, where H_j is the nonempty convex set in $\mathbb{R}$ defined as $H_j = \{x : l_j \leq x \leq u_j\}$. If $j \in B$ in the optimal solution to $\bar{Q}$, then the* concave envelope *taken over H_j is the function $\hat{\phi}_j(x_j)$ such that*

$$\hat{\phi}_j(x_j) = \bar{\phi}_j(x_j) \tag{5}$$

If $j \in \mathrm{NL}$ or $j \in \mathrm{NU}$ in the optimal solution to $\bar{Q}$ then concave envelope *taken over H_j is any function $\hat{\phi}_j(x_j)$ that satisfies the following properties:*

(OB.1) At each point $l_j \leq x_j \leq u_j$, $\hat{\phi}_j(x_j)$ is less than or equal to $\phi_j(x_j)$.

(OB.2) The function $\hat{\phi}_j(x_j)$ is concave for each $j \in J$.

(OB.3) If $j \in \mathrm{NL}$, the reduced cost function $\Delta_j(x_j)$ is monotonically increasing for $x_j \geq l_j$. If $j \in \mathrm{NU}$, the reduced cost function $\Delta_j(x_j)$ is monotonically decreasing for $x_j \leq u_j$.

(OB.4) If $j \in \mathrm{NL}$, then $\Delta_j(x_j) \geq 0$ for $x_j \geq l_j$. If $j \in \mathrm{NU}$, then $\Delta_j(x_j) \geq 0$ for $x_j \leq u_j$.

The (underestimating) concave envelope relaxation of problem Q, denoted $\hat{Q}$, can now be defined as follows:

$$(\hat{Q}) \qquad \min \hat{\phi}(\underline{x}) \text{ s.t. } \underline{x} \in \hat{S} = \bar{X} \cap \hat{H} \tag{6}$$

where $\bar{X}$ is defined in equation (1), and $\hat{H} = \left\{\underline{x} : \hat{\underline{l}} \leq \underline{x} \leq \hat{\underline{u}}\right\}$ with $\hat{\underline{l}} \in \left(\ldots, \hat{l}_j, \ldots\right)^{\mathrm{T}} \in \mathbb{R}^{n+m}$ and $\hat{\underline{u}} \in (\ldots, \hat{u}_j, \ldots)^{\mathrm{T}} \in \mathbb{R}^{n+m}$ where

$$\hat{l}_j = \begin{cases} l_j & \text{if } j \in NL \\ -\infty & \text{if } j \in B \cup NU \end{cases} \qquad \hat{u}_j = \begin{cases} u_j & \text{if } j \in NU \\ +\infty & \text{if } j \in B \cup NL \end{cases} \tag{7}$$

Problem $\hat{Q}$, the concave envelope relaxation of problem Q, is generally referred to as the nonconvex relaxation of problem Q, for reasons that will become apparent subsequently in this paper.

Note that the specification of $\hat{l}_j$, $\hat{u}_j$, and $\hat{\phi}_j(x_j)$ means that problem $\hat{Q}$ exhibits three important properties. First, $\hat{Q}$ is a relaxation of subproblem Q (property (OB.1)). Hence, $\nu\left[\hat{Q}\right] \leq \nu[Q]$. Second, the nonconvex relaxation $\hat{Q}$ is purposely formulated so that its optimal solution corresponds to the optimal solution to the linear relaxation $\bar{Q}$. Thus $\underline{\hat{x}} = \underline{\bar{x}}$, where $\underline{\hat{x}} = (\ldots, \hat{x}_j, \ldots)^{\mathrm{T}} \in \mathbb{R}^{n+m}$ is the optimal solution vector for problem $\hat{Q}$ (properties (OB.2), (OB.3), and (OB.4)). Finally, $\hat{l}_j$, $\hat{u}_j$, and $\hat{\phi}_j(x_j)$ have been defined in such a way that the parametric analysis of problem $\hat{Q}$ is particularly easy to perform.

Three possible specifications of $\hat{\phi}_j(x_j)$ for $j \in NL \cup NU$ are given in the remainder of this Section. The three specifications are called the *linear*, *concave*, and *mixed* formulations, and are identified by the superscripts L, C, and M respectively.

4.1 Linear Formulation

The objective function $\hat{\phi}_j^L(x_j)$ for the linear formulation when $j \in NL \cup NU$ is defined as the linear function

$$\hat{\phi}_j^L(x_j) = \bar{\phi}_j(x_j) \tag{8}$$

4.2 Concave Formulation

In the second, concave, formulation, the specification of the objective function $\hat{\phi}_j^C(x_j)$ depends upon whether $j \in NL$ or $j \in NU$. If $j \in NL$, then

$$\hat{\phi}_j^C(x_j) = \begin{cases} \min\{\phi_j(x_j), \phi_j(u_j) + \bar{z}_j \cdot (x_j - u_j)\} & x_j \leq u_j \\ \phi_j(u_j) + \bar{z}_j \cdot (x_j - u_j) & x_j > u_j \end{cases} \tag{9}$$

where $\bar{z}_j$ is defined previously. If $j \in NU$, then

$$\hat{\phi}_j^C(x_j) = \begin{cases} \min\{\phi_j(x_j), \phi_j(l_j) + \bar{z}_j \cdot (x_j - l_j)\} & x_j \geq l_j \\ \phi_j(l_j) + \bar{z}_j \cdot (x_j - l_j) & x_j < l_j \end{cases} \tag{10}$$

4.3 Mixed Formulation

In the third, mixed, formulation, the objective function $\hat{\phi}_j^M(x_j)$ is specified simply as

$$\hat{\phi}_j^M(x_j) = \max\left\{\hat{\phi}_j^L(x_j), \hat{\phi}_j^C(x_j)\right\} \tag{11}$$

Clearly, $\hat{\phi}_j^M(x_j)$ is a neither concave nor convex (hence problem $\hat{Q}$ is referred to as the nonconvex relaxation of Q). However, note that for any particular variable j, $\hat{\phi}_j^M(x_j)$ can equivalently be specified as *either* $\hat{\phi}_j^L(x_j)$ *or* $\hat{\phi}_j^C(x_j)$, whichever is greater. Thus, since both $\hat{\phi}_j^L(x_j)$ and $\hat{\phi}_j^C(x_j)$ are concave envelopes for $\phi_j(x_j)$, then $\hat{\phi}_j^M(x_j)$ can be treated as though it were also a concave envelope for $\phi_j(x_j)$ even though it is clearly not a concave function.

Figures 2, 3 and 4 show typical representations of $\hat{\phi}_j^M(x_j)$ when $j \in B$, $j \in NL$ and $j \in NU$, respectively.

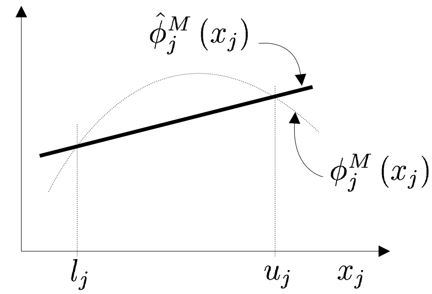

Figure 2: Arc cost function $\hat{\phi}_j^M(x_j)$ for $j \in B$

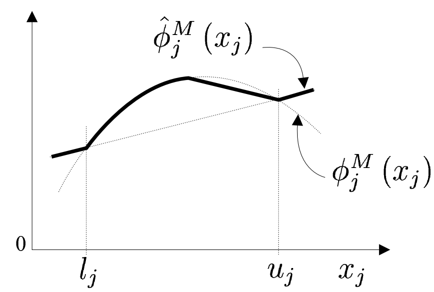

Figure 3: Arc cost function $\hat{\phi}_j^M(x_j)$ for $j \in NL$

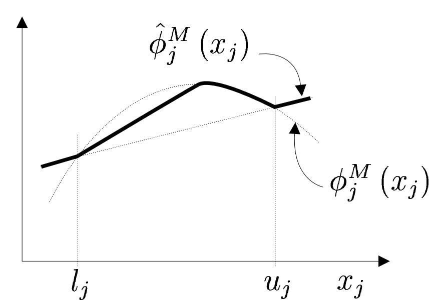

Figure 4: Arc cost function $\hat{\phi}_j^M(x_j)$ for $j \in NU$

5 Post-Optimal Parametric Analysis

The parametric analysis given below describes how the solution to the nonconvex relaxation $\hat{Q}$ changes as the value of a single variable varies from its optimal value in problem $\hat{Q}$. Let $k \in J$ be the index of the arc being altered, and let "$\hat{Q} \mid x_k = \hat{x}_k + \delta_k$" denote problem $\hat{Q}$ augmented with the constraint "$x_k = \hat{x}_k + \delta_k$" where $\hat{x}_j = \bar{x}_j$ and δ_k is a (as yet unspecified) scalar. To describe the parametric analysis of problem "$\hat{Q} \mid x_k = \hat{x}_k + \delta_k$" using δ_k as the parameter, it is convenient to define the *parametric function* $\theta_k(\delta_k)$ as

$$\theta_k(\delta_k) = \nu\left[\hat{Q} \mid x_k = \hat{x}_k + \delta_k\right] - \nu\left[\hat{Q}\right] \tag{12}$$

Note that $\theta_k(\delta_k)$ is a unimodal function with a minimum value of $\theta_k(\delta_k) = 0$ at $\delta_k = 0$. In addition, $\theta_k(\delta_k)$ is nonincreasing for all $\delta_k \leq 0$ and nondecreasing for all $\delta_k \geq 0$.

To express $\theta_k(\delta_k)$ explicitly, it is useful to refer to the reduced cost function, $\Delta_j(x_j)$, associated with $\hat{\phi}_j(x_j)$. The calculation of the parametric function $\theta_k(\delta_k)$ depends upon whether $k \in B$, $k \in NL$, or $k \in NU$.

If $k \in NL$, then changing x_k from $\hat{x}_k$ to $\hat{x}_k + \delta_k$ in problem $\hat{Q}$ means that the minimum objective function value will increase by $\Delta_k(l_k + \delta_k) - \Delta_k(l_k)$ if $\delta_k \geq 0$, and by an infinite amount if $\delta_k < 0$. That is, if $k \in NL$, then

$$\theta_k(\delta_k) = \begin{cases} +\infty & \text{if } \delta_k < 0 \\ \Delta_k(l_k + \delta_k) - \Delta_k(l_k) & \text{if } \delta_k \geq 0 \end{cases} \tag{13}$$

Similarly, if $k \in NU$, then

$$\theta_k(\delta_k) = \begin{cases} \Delta_k(u_k + \delta_k) - \Delta_k(u_k) & \text{if } \delta_k \leq 0 \\ +\infty & \text{if } \delta_k > 0 \end{cases} \tag{14}$$

If $k \in B$, then changing x_k from $\hat{x}_k$ to $\hat{x}_k + \delta_k$ in problem $\hat{Q}$ means that a *single* nonbasic variable x_j will change from $\hat{x}_j$ to $\hat{x}_j - (\delta_k/\bar{a}_{kj})$; and the minimum objective function value will increase by $\Delta_j(\hat{x}_j - (\delta_k/\bar{a}_{kj})) - \Delta_j(\hat{x}_j)$. The reason why the value of only a single nonbasic variable will change in this case is as follows. First, $\hat{l}_j$ and $\hat{u}_j$ for each $j \in N$ are defined such that these capacities will not be binding for any value of $\delta_k \neq 0$. Second, for any given δ_k, each $\hat{\phi}_j(\hat{x}_j - (\delta_k/\bar{a}_{kj}))$ can be treated as a single concave function. Combined, these properties mean that $\Delta_j(x_j)$ can be treated as an uncapacitated concave function.

To describe the calculation of $\theta_k(\delta_k)$ when $k \in B$, we define another function, denoted $\theta_{kj}(\delta_k)$, as

$$\theta_{kj}(\delta_k) \begin{cases} \Delta_j(\hat{x}_j - (\delta_k/\bar{a}_{kj})) - \Delta_j(\hat{x}_j) & \text{if } \delta_k < 0 \text{ and } j \in J_k^- \\ 0 & \text{if } \delta_k = 0 \\ \Delta_j(\hat{x}_j - (\delta_k/\bar{a}_{kj})) - \Delta_j(\hat{x}_j) & \text{if } \delta_k > 0 \text{ and } j \in J_k^+ \\ +\infty & \text{otherwise} \end{cases} \tag{15}$$

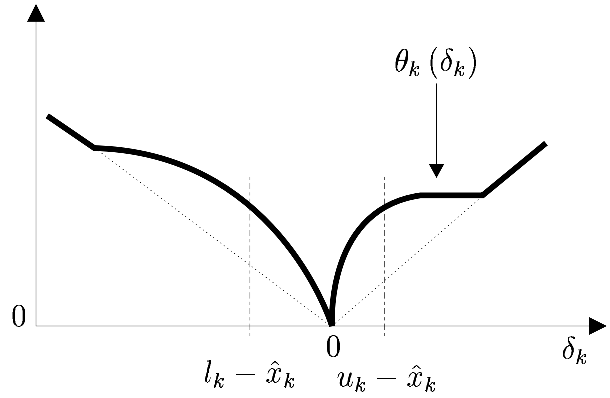

Figure 5: Parametric function $\theta_k(\delta_k)$ for $k \in B$

where the index subsets J_k^+ and J_k^- are defined as

$$J_k^- = \{j : (j \in NL \text{ and } \bar{a}_{kj} > 0) \text{ or } (j \in NU \text{ and } \bar{a}_{kj} < 0)\} \tag{16}$$

$$J_k^+ = \{j : (j \in NL \text{ and } \bar{a}_{kj} < 0) \text{ or } (j \in NU \text{ and } \bar{a}_{kj} > 0)\} \tag{17}$$

Then, if $k \in B$, the function $\theta_k(\delta_k)$ is given by

$$\theta_k(\delta_k) = \min_{j \in J} \{\theta_{kj}(\delta_k)\} \tag{18}$$

Figure 5 shows a typical representation of $\theta_k(\delta_k)$ when $k \in B$.

6 Applications of Concave Envelopes

Notice that, for any $k \in J$, the function $\theta_k(\delta_k)$ given in (13), (14), and (18) is based directly on information available in the solution to $\bar{Q}$. The parametric analysis of the nonconvex relaxation can therefore be used as part of a solution procedure that incorporates the solution of linear programming relaxations of Q (or its subproblems) to obtain information about the solution to the original problem Q. For example, techniques, such as capacity improvement (see, for example, [13]) and penalties (see, for example, [6]) may be used within standard solution algorithms, such as branch and bound, to increase the convergence speed of the parent algorithm. Typically, such techniques make use of post optimal analysis of linear programming relaxations of subproblems of Q. They can therefore be extended or enhanced by instead using post optimal analysis of the nonconvex relaxation presented in this paper. Analysis presented in [1] and [2] indicate significant improvement in solution time (up to 50%) can be expected when nonconvex, rather than linear, relaxations are used in these procedures.

References

[1] G. J. Bell. *Nonconvex Network Optimisation: Solution Methods and Applications.* PhD thesis, University of Canterbury, 1998.

[2] G. J. Bell, B. W. Lamar, and C. A. Wallace. Capacity improvement, penalties, and the fixed charge transportation problem. Submitted to Naval Research Logistics, 1997.

[3] H. P. Benson. Deterministic algorithms for constrained concave minimization: A unified critical survey. *Naval Research Logistics*, 43:765–795, 1996.

[4] J. M. Bloemhof-Ruwaard, M. Salomon, and L. N. van Wassenhove. The capacitated distribution and waste disposal problem. *European Journal of Operational Research*, 88:490–503, 1996.

[5] D. E. Blumenfeld, L. D. Burns, C. F. Daganzo, M. C. Frick, and R. W. Hall. Reducing logistics costs at general motors. *Interfaces*, 17:26–47, 1987.

[6] K. M. Bretthauer. A penalty for concave minimisation derived from the tuy cutting plane. *Naval Research Logistics*, 41:455–463, 1994.

[7] P. C. Fetterolf and G. Anandalingam. A lagrangion relaxation technique for optimizing interconnection of local area networks. *Operations Research*, 40:678–688, 1992.

[8] R. Horst, P. M. Pardalos, and N. V. Thoai. *Introduction to Global Optimization.* Klewer Academic Publishers, Berlin, 1995.

[9] R. Horst and H. Tuy. *Global Optimization: Deterministic Approaches.* Springer-Verlag, Dordrecht, third edition, 1996.

[10] J. J. Jarvis, R. L. Rardin, V. E. Unger, R. W. Moore, and C. C. Schimpeler. Optimal design of regional wastewater systems: A fixed-charge network model. *Operations Research*, 26:538–550, 1978.

[11] J. G. Klincewicz. Solving a freight transport problem using facility location techniques. *Operations Research*, 38:99–109, 1990.

[12] P. C. Kumar. A conceptual decison model for impact analysis of industry regulation: A milp approach. *Computers and Operations Research*, 15:369–379, 1988.

[13] B. W. Lamar. An improved branch and bound algorithm for minimum concave cost network flow problems. *Journal of Global Optimization*, 3:261–287, 1993.

[14] F. J. Vasko, D. D. Newhart, K. L. Stott, and F. E. Wolf. Using a facility location algorithm to determine optimum cast bloom lengths. *Journal of the Operational Research Society*, 47:341–346, 1996.

[15] B. Yaged. Minimum cost routing for static network models. *Networks*, 1:139–172, 1971.