On Symmetric Traveling Salesman Polytope and Separation

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Abstract

The Symmetric Traveling Salesman Problem is posed in Arthanari [?], as a Multistage Insertion problem. The $0 - 1$ programming formulation of the same is given in three subscripted variables Xs. The slack variables us, that arise out of this formulation are precisely the edge-tour incidence vectors. In this paper we consider the LP relaxation of this formulation and show that if the X variables further satisfy certain polyhedral inequalities (called star-inequalities) then u, corresponding to X, is in the convex hull of tours Q_n. Illustrative examples are given. It is shown in a sequel to this work, that given a u in the subtour elimination polytope and a corresponding X feasible to the above said LP relaxation, it can be verified in polynomial time in n, number of cities, whether X satisfies the star-inequalities. Any violated star-inequality can be found otherwise.

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1 Introduction

The Symmetric Traveling Salesman Problem (STSP) is to find a minimum cost tour that starts from the home city and visits every city in a given set of cities once and returns back to the home city. Being one of the typical combinatorial optimisation problems (COP) of the 'Hard' kind, it is extensively studied (Jünger et. al [?]) Dantzig et. al [?] formulated the Asymmetric Traveling Salesman Problem as a 0-1 linear program on a graph (V, E). Their formulation for the symmetric case gives rise to the standard Subtour Elimination Polytope SEP_n. Arthanari [?] posed the Symmetric Traveling Salesman Problem (STSP) as a multistage decision problem and gave a $0 - 1$ programming formulation of the same, involving three subscripted variables. The slack variables that arise out of this formulation are precisely the edge-tour incidence vectors. Multistage decision (dynamic programming) approach to TSP is not new. However the earlier formulations are different from that of [?]. In [?] this formulation is shown to yield the $SEP(n)$. Thereby reducing the number of constraints to a polynomial in n. Recently Carr [?] has given an equivalent relaxation called Cycle Shrink, to the SEP_n. He also uses this relaxation to separate in polynomial time, over classes of STSP inequalities. It is now known that the cycle shrink relaxation is conceptually same as multistage insertion [?].

In this paper we address ourselves to the problem of checking whether a given $u \in SEP_n$ is in the STSP polytope Q_n. In section 2 preliminaries and notations are explained. Section 2.1 deals with MI formulation. The problem of finding maximal flow through a network, eminently studied by Ford and Fulkerson [?], forms an important tool in the subsequent sections. With this in view, section 2.3 addresses Gale's theorem and related issues. Section 3 gives the main results of this paper. Star-inequalities are defined and it is shown that if there exists a X such that $(X; u)$ is feasible for the MI relaxation and X satisfies the star-inequalities then $u \in Q_n$. Illustrative examples are given. The concluding section, Section 5, examines the consequences of the main findings and indicates the direction of further research regarding the complexity of the suggested approach.

2 Notations & Preliminaries

Let n be an integer, $n \geq 3$. Let $V_n = \{1, \ldots, n\}$ be the set of vertices. Let $E_n = \{(i,j) | i,j \in V_n; i < j\}$ be the set of edges. We also call them pairs. Let $K_n = (V_n; E_n)$ denote the complete graph on n vertices.

We denote the elements of E_n by e where $e = (i,j)$. We also use the notation ij for (i,j). For a subset $F \subseteq E_n$ we write the characteristic vector of F by $x_F \in \mathbb{R}^{|E_n|}$ where

$$x_F(e) = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{otherwise} \end{cases}$$

At times, we denote the characteristic vector by $\bar{F}$. If $F \subseteq E_r \subseteq E_n$, $r \leq n$, then $\bar{F} \in \mathbb{R}^{|E_r|}$. We also use $\bar{F}$ to denote the extension of $\bar{F}$ to $\mathbb{R}^{|E_n|}$, by augmenting $\bar{F}$ with zero for every $e \in E_n \setminus E_r$.

Let $G = (V; E)$ be a subgraph of K_n, that is, $V \subseteq V_n$ and $E \subseteq \{(i,j) \in E_n | i,j \in V\}$. Given a Graph $G = (V; E)$, for a subset $S \subseteq V$ we write

$$E(S) = \{ij | ij \in E; i, j \in S\}.$$

Let for any subset S of vertices of V, $\delta(S)$ denote the set of edges in E with one end in S and the other in $S^c = V \setminus S$.

We say $i \in V$ has degree r if cardinality of $\delta(i) = r$.

A subset H of E is called a Hamiltonian Cycle in G, in case, it is the edge set of a simple cycle in G, of length $|V|$. Or $G = (V; H)$ is a subgraph of G which is connected and each vertex has degree 2. We also call such a hamiltonian cycle a $|V| -$ tour, in G. Given $u \in \mathbb{R}^{|E_n|}$, $F \subseteq E_n$, we define,

$$u(F) = \sum_{e \in F} u(e) = \sum (u(e) : e \in F)$$

Let Q_n denote the Symmetric Traveling Salesman Polytope, STSP, given by

$$Q_n = conv(H)$$

where H denotes the set of all Hamiltonian cycles in K_n or $n -$ tours.

2.1 MI Formulation

Let $p_k = k(k-1)/2$, for $k \in V_n$. Let $\tau_n = \sum_{k=4}^{n} p_{k-1}$.

Let $x_k(e)$ be defined $\forall e \in E_{k-1}$ and $\forall k \in V_n \setminus V_3$.

Let X denote $(x_{12,4}, \ldots, x_{n-2\,n-1,n}) \stackrel{def}{=} (x_4, \ldots, x_n) \in R^{\tau_n}$.

Here $x_k = (x_{12,k}, \ldots, x_{k-2\,k-1,k})$, $\forall k \in V_n \setminus V_3$.

Let $C_{ijk} = C_{ik} + C_{jk} - C_{ij}$, where C is defined for each $e \in E_n$, the cost of travel, using e. Let C denote the vector of C_{ijk} for $1 \le i < j < k$ & $4 \le k \le n$.

Let $u \in R^{|E_n|}$ be defined for each $e \in E_n$.

We have the following formulation from [?], which will be called the Multistage-insertion formulation (MI formulation for short):

$$\text{minimize} \quad CX \tag{1}$$

subject to

$$x_k(E_{k-1}) = 1 \quad \forall k \in V_n \setminus V_3 \tag{2}$$

$$\sum_{k=4}^{n} x_k(e) + u_e = 1 \quad \forall e \in E_3 \tag{3}$$

$$-x_j(\delta(i) \cap E_{j-1}) + \sum_{k=j+1}^{n} x_k(e) + u(e) = 0 \quad \forall e = (i,j) \in E_{n-1} \setminus E_3 \tag{4}$$

$$-x_n(\delta(i) \cap E_{n-1}) + u(e) = 0 \quad \forall e = (i,n) \in \delta(V_{n-1}) \tag{5}$$

$$X \in B^{\tau(n)} \quad \& \quad u \in R_{+}^{|E_n|}. \tag{6}$$

Remark: 1

- MI formulation models the following multistage insertion. At the start we have the unique 3-tour given by {(1,2),(1,3),(2,3)}. We choose at the $(k-3)^{rd}$ stage, a pair from the available pairs in the $(k-1)$-tour, resulting from the earlier decisions. We have an n-tour at the last stage. The cost of these decisions is reflected by the objective function, which gives the total incremental cost of the insertions made.
- (??) ensures with $0-1$ restriction on X that each k is inserted in exactly one pair.
- (??) ensures that each of the pairs (1,2), (1,3) and (2,3) are used for insertion by utmost one k.
- (??) ensures that each of the other pairs (1,4) to $(n-2, n-1)$ is used for insertion only when they are available.
- (??) defines $u(e)$ for each of the pairs $(1,n)$ to $(n-1,n)$, resulting from all the insertions made in stages 1 through $n-3$.
- relaxing the integrality restriction in (??), we get the MI Polytope (P_n), given by the feasible solutions to the relaxed problem.

It has been shown that this formulation is equivalent to the DFJ formulation, in the sense that the $u(e); e \in E_n$ feasible for the relaxation of MI formulation is feasible for the SEP$_n$ and given a feasible solution to the DFJ relaxation, we have an $(X; u)$ feasible for MI formulation ([?]). Notice that the u variables are the slack variables in this formulation. And minimising $C(u)$ is equivalent to minimising CX, as can be seen from the definition of C.

In general, let $E_{[n]}$ denote the matrix corresponding to equation (??); let $A_{[n]}$ denote the matrix corresponding to the equations (?? to ??). Let $A^{(n)}$ denote the submatrix of $A_{[n]}$ corresponding to x_n. Let 1_r denote the row vector of r 1's. Then we can write recursively

$$E_{[n]} = \begin{pmatrix} 1_{p_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1_{p_{n-2}} & 0 \\ 0 & 0 & \cdots & 1_{p_{n-1}} \end{pmatrix} = \begin{pmatrix} E_{n-1} & 0 \\ 0 & 1_{p_{n-1}} \end{pmatrix};$$

and

$$A_{[n]} = \begin{pmatrix} I_3 & \cdots & I_{p_{n-2}} & I_{p_{n-1}} \\ -M_3 & \cdots & -M_{n-2} & -M_{n-1} \\ \vdots & \cdots & 0 & \\ 0 & & & \end{pmatrix} = \begin{pmatrix} A_{[n-1]} & I_{p_n} \\ 0 & -M_n \end{pmatrix}$$

where M_i is the $i \times p_i$ node-pair incidence matrix.

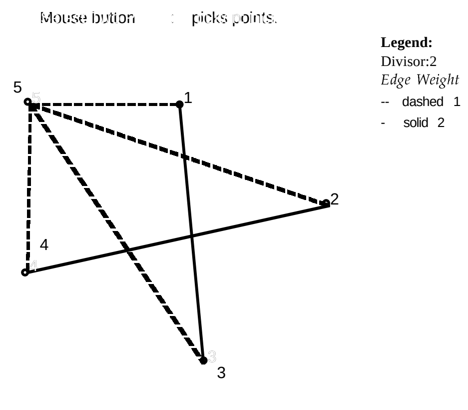

Figure:1. Support Graph of u.

Example: 2.1 Consider $n = 5$, then $p_4 = 6$; $p_5 = 10$; and $\tau_n = 9$. Let X be given by,

$$X' = (1/2\ 0\ 1/2;\ 1/2\ 0\ 0\ 0\ 0\ 1/2) \in R^{\tau_5}.$$

where X' denotes transpose of X.

X satisfies the equality restriction (??) of the MI relaxation. Also, the corresponding u obtained from the rest of the equations (?? - ??) is, 0. We have,

$$u' = \begin{pmatrix} e_3 \\ 0 \end{pmatrix} - A_{[5]}X = (0\ 1\ 1/2\ 1/2\ 1\ 0\ 1/2\ 1/2\ 1/2\ 1/2).$$

So (X, u) is a feasible solution to the MI relaxation, that is, it is in F_5. The support graph of u is shown in figure 1, where the edge weights are shown after multiplying by 2.

2.2 Gale's Requirements and FAT Problem

Let $N = (V, A)$ be an arbitrary directed network with node set V, and set of directed arcs A with a nonnegative capacity function $c : A \to R_+$. Let O and D be distinct subsets of V; O is the set of sources, D is the set of sinks. For each $\alpha \in O$, we have a nonnegative number a_α, called the supply at α. Similarly, for each $\beta \in D$ we have the demand at β given by a nonnegative number b_β. Let a non negative number $f_{\alpha\beta}$, called the flow along arc $(\alpha, \beta) \in A$ be defined for each arc. Let $out(\alpha) = \sum_{(\alpha,\beta)\in A} f_{\alpha,\beta}$ & $in(\alpha) = \sum_{(\beta,\alpha)\in A} f_{\beta,\alpha}$. Actually $in(\,)$, $out(\,)$ are defined for a given f, but we do not explicitly show this dependance in this notation. Let $a(S)$ denote the cumulated supply over subset $S \subseteq O$ and $b(S)$ is defined similarly for a subset $S \subseteq D$. For any subset L of V we denote by $c(L, \bar{L})$ the cumulated capacity of arcs from L to $\bar{L}$.

The following theorem states the necessary and sufficient conditions for existance of a feasible flow in the network satisfying all demands and not exceeding any supply.

Theorem 2.1 (Gale's Supply & Demand Theorem) $\exists$ an f satisfying

$$0 \le f_{\alpha,\beta} \le c_{\alpha,\beta} \quad \forall (\alpha, \beta) \in A \tag{7}$$

$$in(\alpha) - out(\alpha) = \begin{cases} -a_\alpha, & \alpha \in O \\ b_\alpha, & \alpha \in D \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

if and only if

$$b(D \cap \bar{L}) - a(O \cap \bar{L}) \le c(L, \bar{L}) \quad \forall L \subseteq V. \tag{9}$$

We shall call the above inequalities (??), Gale Requirements. See ([?]) for a proof of this theorem.

We now consider a special case when $V = O \cup D$ & $A = \{(\alpha, \beta) | \alpha \in O$ and $\beta \in D\}$; and for each $\alpha \in O$, we define, $F(\alpha) = \{(\alpha, \beta) \in A |$ for some $\beta \in D\}$ called the forbidden arcs from α. $F(\alpha) \neq \emptyset$, $\forall \alpha \in O$. Let $F = [\cup_{\alpha \in O} F(\alpha)$, denote the set of forbidden arcs. The following problem is called the forbidden arc transportation problem.

(FAT Problem):
Find $f_{\alpha\beta} \ge 0$ $\forall (\alpha, \beta) \in A \setminus F$ such that
$out(\alpha) = a_\alpha$; $\alpha \in O$ & $in(\beta) = b_\beta$; $\beta \in D$.
FAT problem is a network feasibility problem, with $c : A \to R_+$ given by

$$c_{\alpha\beta} = \begin{cases} 0 & \text{if } (\alpha, \beta) \in F(\alpha) \\ \infty & \text{otherwise} \end{cases}$$

where ∞ is sufficiently large, say $\infty = \max[\sum_{\alpha \in O} a(\alpha), \sum_{\beta \in D} b(\beta)]$. This choice of capacity function ensures that there is no flow through arcs belonging to F and there is no real capacity restriction on the flow through other arcs.

3 Main Results

Let H_k denote the set of all hamiltonian cycles in $K_k = (V_k, E_k)$, for $3 \le k \le n-1$. We call them, as mentioned earlier, $k-$tours. Every integer solution to MI formulation gives a hamiltonian cycle in H and vice versa. Recall that F_n denotes the set of (X, u) feasible for the MI relaxation. Let (X/k) be the restriction of X up to the $(k-3)^{rd}$ stage, that is $(X/k) = (x_4, \ldots, x_k)$ for $4 \le k \le n$; for a given (X, u) feasible. Let $U^{(k-3)}$ denote the consequence of the decisions made in (X/k). This can be obtained from the MI relaxation for $n = k$ by substituting (X/k) in the equations and solving for the slack variables. Given a T in H_k, for $k \in V_{n-1} \setminus V_3$, we denote by $D(T)$, the corresponding $\{e_l \in E_{k-1} | 4 \le l \le k$ such that $x_l(e_l) = 1\}$, where X is the integer solution corresponding to T. $D(T)$ is called the cause of T. That is, $(X, \bar{T}) \in F_k$.

Given $(X, u) \in F_n$; $k \in V_{n-1} \setminus V_3$, and $T \in H_k$; we define

$$m_k(X, T) = \min_{4 \le l \le k} \{x_l(e_l) : e_l \in D(T)\}.$$

Let $m_k(X, S) = \sum_{T \in S} m_k(X, T)$. Let $\underline{S} = \{e | e \in T$ for some $T \in S\}$. $\underline{S}$ denotes the set of pairs covered by S.

Remark: 2

- † Loosely speaking $m_k(X, T)$ overestimates the amount of "weight attached" by (X/k) to a $T \in H_k$. If $U^{(k-3)}$ is a convex combination of some $k-$tours and T is one among them, then $m_k(X, T)$ is positive, otherwise, it is 0.
- † Let X correspond to an $n-$tour then X is an integer vector and so (X/k) gives rise a $k-$tour. So if T is that $k-$tour then $m_k(X, T) = 1$. Also, in this case, if $T \in S$, then $m_k(X, S) = 1$ as well. This is so because, $m_k(X, T') = 0 \; \forall \; T' \ne T$ and $T' \in H_k$.

Lemma 1 Given $(X, u) \in F_n$, consider x_4 and the corresponding $U^{(1)}$. We have

$$U^{(1)} = [\bar{T}_1 \bar{T}_2 \bar{T}_3] x_4,$$

where $D(T_1) = e_1 = \{1, 2\}$, $D(T_2) = e_2 = \{1, 3\}$, and $D(T_3) = e_3 = \{2, 3\}$.

Recall that $\bar{T}$ stands for the characteristic vector of T and x_k denotes the vector $(x_{12,k}, \ldots, x_{k-2\,k-1,k})$.

This lemma shows that for every feasible choice of x_4, we have, the consequent $U^{(1)}$ is $\in conv(H_4)$. The next lemma gives the result on the star-inequalities for $n = 5$.

Lemma 2 Given $(X, u) \in F_5$. If X satisfies for $k = 4$,

$$m_k(X, T) \le x_{k+1}(T) = \sum_{e \in T} x_{k+1}(e) \quad \text{for all } T \in H_k$$

then $u \in Q_5$.

This lemma forms the basis for the main theorem. A straightforward proof is attempted here, exploiting the simplicity of the FAT problem related to $n = 5$. The inequality $m_k(X, T) \geq x_{k+1}(T)$, is called the star inequality. Star inequality for each $T \in H_4$ is sufficient in this case. In general, as can be seen from the main theorem, we require star inequalities to be satisfied for every subset $S \subseteq H_k$ $\forall k \in V_{n-1} \setminus V_3$, for ensuring $u \in Q_n$.

Proof of Lemma ??: Consider $(X, u) \in F_n$ & X satisfying

$$x_4(D(T)) \stackrel{def}{=} m_4(X, T) \geq \sum_{e \in T} x_5(e) \;\; \forall\, T \in H_4. \quad (10)$$

Look at the FAT problem as defined below:

Find $f_{\alpha\beta}$ such that,

$$\sum_{(\alpha,\beta) \in A} f_{\alpha\beta} = x_4(D(T_\alpha)) = x_4(e_\alpha) \stackrel{def}{=} a_\alpha, \; \alpha \in O = \{1, 2, 3\}$$

$$\sum_{(\alpha,\beta) \in A} f_{\alpha\beta} = x_5(e_\beta) \stackrel{def}{=} b_\beta, \; \beta \in D = \{1, \ldots, 6\}$$

where $A = \{(\alpha, \beta) | \alpha \in O, \beta \in D\}$ and $F(1) = \{(1,1), (1,6)\}$, $F(2) = \{(2,2), (1,5)\}$ and $F(3) = \{(3,3), (1,4)\}$. Assuming zero capacity for the arcs in F, and using a_α, b_β for supply and demand at α and β, we have the total availability equal to total demand, namely 1. This follows from feasibility of X (see equation (??)).

Consider the network, obtained by adding a supersource, Θ, with an arc (with capacity a_α) connecting α in O and similarly a supersink, $\mathcal{K}$, is added, with an arc from β in D, (with capacity b_β). The maximal flow through this network is 1 ( ) the FAT problem is feasible.

Claim: The maximal flow through the network is indeed 1.

Hence we have a feasible f for the FAT problem under consideration. Now consider the $U^{(1)}$ resulting from x_4, as given by lemma ??.

Let M denote the matrix corresponding to the T_α, $\alpha \in O$ as columns in that order. We have $M x_4 = U^{(1)}$. From the feasible solution to the FAT problem we have $f_{\alpha\beta} \geq 0$ and add up to 1. Restrict the attention to (α, β) such that $f_{\alpha\beta} > 0$. We have,

$$T_\alpha - A^{(5)}_\beta = T_{\alpha\beta}; \quad (11)$$

where, $T_{\alpha\beta}$ is the 5 − tour obtained from $T_\alpha \in H_4$ by inserting 5 in an e_β in T_α. And $A^{(5)}$ is as defined earlier, $A^{(5)}_\beta$, refers to the column corresponding to $x_5(e_\beta)$.

From feasibility, that is, $(X, u) \in F_n$, $n = 5$,

$$M x_4 - A^{(5)} x_5 = u. \quad (12)$$

Replacing x_4 and x_5 in this expression by the equivalent expressions in $f_{\alpha\beta}$, we obtain,

$$M \begin{pmatrix} \sum_\beta f_{1\beta} \\ \vdots \\ \sum_\beta f_{3\beta} \end{pmatrix} - A^{(5)} \begin{pmatrix} \sum_\alpha f_{\alpha 1} \\ \vdots \\ \sum_\alpha f_{\alpha 6} \end{pmatrix} = u. \quad (13)$$

Consider the columns of M and $-A^{(5)}$ having same coefficient $f_{\alpha\beta}$. Combining together we get $T_{\alpha\beta}$.

Thus,

$$\sum_{(\alpha;\beta);f_{\alpha\beta}>0} f_{\alpha\beta} T_{\alpha\beta} = u.$$

Therefore we have expressed u as a convex combination of some $T \in H_5$. Hence, $u \in Q_5$ as required. (Ushakin [?], obtained this result by computation.)

Proof of the Claim: Suppose the maximal flow from 0 to ϕ is $< \infty$. then $\exists\, \alpha_0 \in O$ such that $f_{0\alpha_0} < a_{\alpha_0}$. Using Ford and Fulkerson's labeling algorithm, α_0 is labelled with $[0^+;\mu = a_{\alpha_0} - f_{0\alpha_0} > 0]$. Every β such that $(\alpha_0;\beta) \not\in F(\alpha)$ is labelled with $[\alpha_0^+;\mu]$. Notice that $(\beta;\phi)$ is saturated, otherwise ϕ will be labelled, contradicting the optimality of the flow. Since by the given condition (s??) on X, for any $\alpha \in O$

$$a_\alpha \le \sum_\beta \{b_\beta \mid (\alpha;\beta) \not\in F(\alpha)\}.$$

So $\exists\, \alpha_1 \neq \alpha_0$ such that $f_{\alpha_1\beta_0} > 0$ for some β_0 labelled from α_0. This implies α_1 is labelled with $[\beta_0^-;\hat\mu = \min(\mu; f_{\alpha_1\beta_0})]$. Notice that every β with $(\alpha_0;\beta) \in F(\alpha_0)$ is such that for all other $\alpha \neq \alpha_0$, $(\alpha;\beta) \not\in F(\alpha)$. In other words, for each $\beta \in D$, there is only one $\alpha \in O$ such that $(\alpha;\beta)$ is forbidden. Thus, every β, with $(\alpha_0;\beta) \in F(\alpha_0)$ can be labelled with $[\alpha_1;\hat\mu]$. So, every node in D is labelled either from α_0 or α_1. At least one of the $\beta \in D$ is such that $(\beta;\phi)$ is not saturated, since the maximal flow is assumed to be $< \infty$. So ϕ can be labelled, leading to a contradiction. Hence, the maximal flow in the network is ∞, as claimed.

Example: 3.1 Consider Example ??. From $X = (x_4;x_5)$, we have,

$$U^{(1)} = \begin{pmatrix} e_3 \\ 0 \end{pmatrix} - A_{[4]} x_4 = (1/2\ 1\ 1/2\ 1/2\ 1\ 1/2)^0.$$

Notice that $U^{(1)}$ is in H_4. Star-inequalities for $n = 5$ require the following conditions to be satisfied by X:

$$x_4(D(T)) \le \sum_{e \in T} x_5(e)\ \forall\, T \in H_4.$$

Since, $x_4((1;3))$ is 0, it is sufficient to verify the condition for $D(T) =$ either $(1;2)$ or $(2;3)$. Now, $x_4((1;2)) = 1/2$ and $x_5(T) = 0$ with $T = \{(1;3);(2;3);(1;4);(2;4)\}$. Or the condition is not satisfied. But, for $x_4((2;3)) = 1/2$ and $x_5(T) = 1$ with $T = \{(1;2);(1;3);(2;4);(3;4)\}$, it is satisfied. Hence we are not able to decide the fate of u.

However, it is easy to verify that the solution X' given below produces the same u, and $(X';u) \in H_5$.

$$X' = (1\,0\,0;0\,0\,1/2\,1/2\,0\,0) \in R^{\text{¿5}}.$$

It is sufficient to verify the conditions for $(1;2)$. We have, $x_4((1;2)) = 1$ and $x_5(T) = 1$ with $T = \{(1;3);(2;3);(1;4);(2;4)\}$, satisfying the condition as an equality. This implies $u \in Q_5$. Indeed, $u = 1/2[T_1 + T_2]$, where,

$$T_1 = \{(1;3);(2;3);(2;4);(1;5);(4;5)\} \text{ and } T_2 = \{(1;3);(1;4);(2;4);(2;5);(3;5)\}.$$

Theorem 3.1 (Main Theorem) Given $(X; u) \in F_n$, if X satisfies

$$m_k(X; S) \leq x_{k+1}(\underline{S}), \qquad (14)$$

$\forall k \in V_{n-1} \setminus V_3$ and $\forall S \subseteq H_k$, then $u \in Q_n$.

Proof:

The proof is by induction on number of cities, n.

Basis for induction: The result is true for $n = 5$, from lemma ??.

Assume the result is true upto $n - 1$. We shall show that the result is true for n.

Consider $(X; u) \in F_n$.

Look at $U^{(n-4)}$, obtained from $(X = n - 1) = (x_4; \ldots; x_{n-1})$.

Now $(X = n - 1; U^{(n-4)}) \in F_{n-1}$ from feasibility of $(X; u)$. Also $U^{n-1} \in Q_{n-1}$ by induction hypothesis.

$\Rightarrow \exists$ a $S_\lambda \subseteq H_{n-1}$ such that $U^{(n-4)} = \sum_\alpha \lambda_\alpha T_\alpha$ with $\lambda_\alpha > 0$, $\sum_\alpha \lambda_\alpha = 1$ & $T_\alpha \in S_\lambda$.

Now cosider the FAT problem with:

O — sources : $\alpha = 1; \ldots; p$; where $p = |S_\lambda|$.

D — sinks : $\beta = 1; \ldots; q$; where $q = |E_{n-1}|$.

a_α — supply : $a_\alpha = \lambda_\alpha$ at α.

b_β — demand : $b_\beta = x_n(e_\beta)$ at β.

$c_{\alpha\beta}$ — capacity : $c_{\alpha\beta} = \begin{cases} 1 & \text{if } (\alpha; \beta) \text{ is not forbidden, that is } e_\beta \in T_\alpha \\ 0 & \text{otherwise.} \end{cases}$

Find $f_{\alpha\beta} \geq 0; \forall \alpha \in O; \beta \in D$, feasible.

From requirements (??)

$\exists$ a feasible $f(\,) \quad \forall L \subseteq O \cup D;\ b(D \cap \bar{L}) - a(O \cap \bar{L}) \leq c(L; \bar{L})$; where $\bar{L} = O \cup D - L$ and $c(L; \bar{L}) = \sum_{i \in L; j \in \bar{L}} c_{ij}$. But here $c(L; \bar{L}) = c(L \cap O; D \cap \bar{L})$. If L is such that $(L \cap O; D \cap \bar{L})$ contains an arc $(\alpha; \beta)$ not forbidden then $c(L \cap O; D \cap \bar{L}) \geq 1$. But $b(D) = 1$, $b(D \cap \bar{L})$ & $a(O \cap \bar{L}) \leq 1$. Therefore Gale's requirements (??) for such L are trivially satisfied.

So, let us assume that $(L \cap O; \bar{L} \cap D)$, then consists of only forbidden arcs. Therefore $c(L \cap O; \bar{L} \cap D) = 0$. $\Rightarrow$ We require $b(D \cap L) \geq a(O \cap L)$. Now $b(D \cap L) = \sum_{\beta \in D \cap L} x_n(e_\beta)$ & $a(O \cap L) = \sum_{\alpha \in O \cap L} \lambda_\alpha$. From our assumption on L, for every $\alpha \in O \cap L$, all the non forbidden $(\alpha; \beta)$ arcs end in $\beta \in (L \cap D)$. Let $J = \{e_\beta \mid e_\beta \in T_\alpha$ for some $\alpha \in O \cap L\}$. Notice that $\lambda_\alpha \leq m_{n-1}(X; T_\alpha)\ \forall \alpha$.

Thus,

$$\sum_{\alpha \in O \cap L} \lambda_\alpha \leq \sum_{\alpha \in O \cap L} m_{n-1}(X; T_\alpha) \leq \sum_{e_\beta \in J} x_n(e_\beta) = \sum_{\beta \in D \cap L} x_n(e_\beta).$$

Hence requirements (??) are met. Thus we have a feasible f for the FAT problem. This means f is a non negative vector adding upto one, as maximum flow in the network $= b(D) = 1$. The rest of the proof, is similar to the that of lemma ??.

Hence the theorem.

Notice that, it is sufficient to check that the FAT problems defined for each k is feasible for a given X, to declare $u \in Q_n$.

4 Conclusions

This paper identifies certain polyhedral inequalities called star-inequalities, that are useful in checking whether a given $u \in SEP_n$ is in Q_n as well. It is interesting to device an algorithm using these star-inequalities to answer the separation question. It is shown, in a sequal to this work, that given a u in the subtour elimination polytope and a corresponding X; such that $(X; u)$ is feasible to the MI relaxation, it can be verified in polynomial time in n, number of cities, whether the Gale's requirements, for each of the FAT problems involved, are met by X. Otherwise a violated star-inequality can be found. This uses the algorithm for finding a minimal witness for infeasibility, developed in [?]. It is also noticed that, any violated star-inequality provides a closed convex set K, such that K^o; the interior of K contains no integer $x' \in Q_n$, but $X \in K^o$: This provides a basis to identify a most violated (with respect to X) valid inequality for Q_n: Further research on the complexity issues and practical computational schemes are on.

Acknowledgments

Prof. David Ryan, Mr. Jim Greenslade and Ms. Sarah Devalian, are thanked for their readiness to sort out my problems with resources, computing and secretarial assistance, respectively. Support from NZISS, NZES is acknowledged.

References

[1] C.C. Aggarwal, R.K. Ahuja, J. Hao, J.B. Orlin, Diagnosing Infeasibilities in Network Flow Problems, Math. Programming 81 (1998) pp 263-280.

[2] T.S. Arthanari, On the Traveling Salesman Problem, Presented at the XI Symposium on Mathematical Programming held at Bonn, West Germany, (1982).

[3] T.S. Arthanari and M. Usha, Equivalence of U_n and $SEP(n)$ and its implications, Preprint, communicated for publication, (1997).

[4] T.S. Arthanari and M. Usha, On the Equivalence of the Multistage-Insertion and Cycle Shrink Formulations of the Symmetric Traveling Salesman Problem, Preprint, (1997).

[5] B. Carr, Separating Over Classes of TSP Inequalities Defined by 0 Node-Lifting in Polynomial Time, Preprint, (1996).

[6] G.B. Dantzig, D. Fulkerson, and S. Johnson, Solution of a Large Scale Traveling Salesman Problem, Operations Research 2 (1954) pp 393-410.

[7] L.R. Ford, D.R. Fulkerson, Flows in Networks, Princeton Univ. Press, (1962).

[8] M. Jũnger, G. Reinelt, and G. Rinaldi, The Traveling Salesman problem, Handbooks in OR & MS, 7, Elsevier Science B.V. (1995).

[9] M. Usha, New inequalities for $n = 5$, private communication, (1998).