On an Application of Pedigree Approach to Symmetric Traveling Salesman Problem

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Abstract

Arthanari[3] gave an alternative formulation for the symmetric traveling salesman problem($STSP$), based on multi stage insertion decisions. In a recent paper Arthanari and Usha[7] study some of the properties of this formulation. The classical STSP polytope is generally studied as embedded in the standard sub tour elimination polytope(SEP). Boyd and Pulleyblank[9] study the structure of the vertices of SEP and identify two classes of fractional vertices to show how complex they can be. Arthanari[4] introduced the objects called pedigrees which are in 1-1 correspondence with the tours. The convex hull of these pedigrees yields a new polytope, called the pedigree polytope. In this paper, we apply some of the necessary and sufficient conditions for membership in pedigree polytope developed in Arthanari[4], to show that some vertices of SEP identified by Boyd and Pulleyblank[9] are not in the corresponding $STSP$ polytope.

1 Introduction

The traveling salesman problem (TSP) is about finding a minimum cost tour that starts from the home city and visits every city once and returns back to the home city. When the cost of traveling from city i to city j is same as that of traveling from j to i, we have the symmetric traveling salesman problem ($STSP$). TSP is one of the typical $\mathcal{NP}$-Hard combinatorial optimisation problems, and has been extensively studied [11]. The seminal paper by Dantzig et.al. [10] formulates the asymmetric traveling salesman problem (ATSP) as a 0-1 linear program on a graph. The analogous formulation for the $STSP$, when the integrality constraints are relaxed, results in the the standard Subtour Elimination Polytope, SEP_n, where n denotes the number of cities. Boyd and Pulleyblank in [9] study optimisation over SEP_n. Optimisation over SEP_n is a first step in solving $STSP$ and it is known that it can be done in polynomial time.(See references in [9].)

Arthanari [3] posed the $STSP$ as a multistage decision problem and gave a $0-1$ programming formulation of the same, involving variables with three subscripts. We refer to this formulation as the Multistage Insertion (MI) formulation. The multistage decision (dynamic programming) approach to TSP is not new. However the earlier formulations are different from that of [3]. The slack variables that arise out of this formulation are precisely the edge-tour incidence vectors. The set of feasible slack variable vectors of the MI - relaxation is shown to be equivalent to SEP_n, in [6]. The number of constraints in the MI-formulation is $(n-3)+n(n-1)/2$ and there are $O(n^3)$ variables. (See [7] for MI- formulation and the motivation for the multistage insertion approach.)

In [4] the author introduced the basic objects called *pedigrees* which are in 1-1 correspondence with the tours. Pedigrees represent the multistage insertion decisions. The convex hull of these pedigrees yields a new polytope, called the *pedigree* polytope. Purpose of studying these polytopes, is to gain new insights into the STSP. In particular, we study the pedigree polytope as embedded in MI-relaxation polytope. Feasibility checking in a transportation problem with certain arcs forbidden is called a FAT problem. A sequence of FAT problems are defined for a solution X to the MI-relaxation for the purpose of obtaining necessary and sufficient conditions for membership in the pedigree polytope.

In this paper we study the problem of finding X that is feasible for MI- relaxation, given u, a fractional solution from the standard subtour elimination polytope, SEP_n. From the equivalence of SEP_n and the set of slack variable vectors, it is easy to see that this problem is finding a feasibile X for MI-relaxation with the slack variables fixed at the given u. However the recursive structure of the MI-relaxation can be exploited while solving this problem. Also the sub problems that need to be solved have a special structure similar to that of a fractional capacited perfect b-matching problem. Capacited perfect b-matching can be defined as a feasible solution to the following problem.

Problem 1.1 *[Capacited perfect b-matching problem] Given a graph* $G = (V, E)$*, and* $b(v) \geq 0$*, integer for all* $v \in V$*, and* $d(e) \geq 0$*, integer for all* $e \in E$*, find* $y(e) \geq 0$*, integer for all* $e \in E$ *such that*

$$y(\delta(v)) = b(v) \quad \text{for all } v \in V \tag{1}$$

$$y(e) \leq d(e) \quad \text{for all } e \in E \tag{2}$$

where $\delta(v) = \{e | e \text{ is an edge incident at } v\}$.

In general various objectives are used for selecting a *best* capacited perfect b-matching, depending on the weight function defined over the edges. Anstee [2] gives a three stage method for solving b-matching problems in strongly polynomial time. The first stage solution however is a fractional solution to the problem. We prove some uniqueness results on the sub problem and use them subsequently. We illustrate the use of the FAT problems developed, to show that the extreme points from Boyd and Pulleyblank classes for $n = 10, 12$ are not in the respective $STSP$ polytope. The steps used in these examples may not be tractable in general. Current research is directed towards developing an algorithm to check membership in the pedigree polytope based on the FAT problems introduced.

Section 2 of this paper deals with the preliminaries and notation along with some required results from related work by the author. For graph related terms see [8] and [1]. The special structure of the MI-relaxation and the other results relating to the sub problems are studied in Section 3. Illustrative examples are given in Section 4.

2 Preliminaries and Notation

Let n be an integer, $n \geq 3$. Let $V_n = \{v_1, \ldots, v_n\}$ be a set of *vertices*. Assuming, without loss of generality, that the vertices are numbered in some fixed order, we write, $V_n = \{1, \ldots, n\}$. Let $E_n = \{(i,j) | i,j \in V_n, i < j\}$ be the set of *edges / pairs*. The cardinality of E_n is denoted by $p_n = n(n-1)/2$. Let $\tau_n = \sum_{k=4}^{n} p_{k-1}$. Let $\tau_n^{'} = \tau_n - (n-3)$. We denote the elements of E_n by e where $e = (i,j)$. We also use the notation ij for (i,j).

Let $X = (\mathbf{x}_4, \ldots, \mathbf{x}_n) \in R^{\tau^n}$, where $\mathbf{x}_k \in R^{p_{k-1}}$, $k \in \{4, \ldots, n\}$. Let $X/k = (\mathbf{x}_4, \ldots, \mathbf{x}_k)$. Let $u \in R^{p_k}$. Given the cost of traveling between i and j, C_{ij}, let $\mathcal{C}_{ijk} = C_{ik} + C_{jk} - C_{ij}$ be the incremental cost for inserting k in (i,j). With the view to understand the recursuive structure of the MI-relaxation, the following definition is introduced.

Definition 2.1 *Let $\mathbf{1}_r$ denote the row vector of r $1'$s. In general, let $E_{[n]}$ and $A_{[n]}$ be defined as below:*

We can write recursively,

$$E_{[n]} = \begin{pmatrix} \mathbf{1}_{p_3} & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & \mathbf{1}_{p_{n-2}} & 0 \\ 0 & 0 & \ldots & \mathbf{1}_{p_{n-1}} \end{pmatrix} = \begin{pmatrix} E_{[n-1]} & 0 \\ \mathbf{0} & \mathbf{1}_{p_{n-1}} \end{pmatrix}.$$

To derive a similar expression for $A_{[n]}$ we first define

$$A^{(n)} = \begin{pmatrix} I_{p_{n-1}} \\ -M_{n-1} \end{pmatrix}$$

where M_i is the $i \times p_i$ node-pair incidence *matrix.*

Then

$$A_{[n]} = \begin{pmatrix} A^{(4)} & | & A^{(5)} & | & & | & A^{(n)} \\ & | & & | & \ddots & | & \\ \mathbf{0} & | & \mathbf{0} & | & & | & \end{pmatrix} = \begin{pmatrix} A_{[n-1]} & | & A^{(n)} \\ \mathbf{0} & | & \end{pmatrix}.$$

Now we can state the MI-relaxation [3], equivalently as given below:

Problem 2.2

$$\begin{aligned} \textit{minimise } \mathcal{C}X & & & \quad (3)\\ \textit{subject to} & & & \\ E_{[n]}X &=& \mathbf{1}_{n-3} & \quad (4)\\ A_{[n]}X+u &=& \begin{pmatrix} \mathbf{1}_3 \\ \mathbf{0} \end{pmatrix}. & \quad (5)\\ X &\geq& 0 \quad \textit{and} & \quad (6)\\ u &\geq& 0. & \quad (7) \end{aligned}$$

The set of feasible solutions to Problem 2.2 is denoted by $\mathcal{F}_n$. Any integer solution (X, u) in $\mathcal{F}_n$ gives a pedigree and its corresponding tour respectively. The objective function minimises the incremental cost of the insertions made according to X. (See [7] for details of MI-formulation.) Let P_n denote the set of integer X such that $(X, u) \in \mathcal{F}_n$. The convex hull of P_n gives us the pedigree polytope $conv(P_n)$. We denote the $STSP$ polytope for n by Q_n.

In [4] for a given $(X, u) \in \mathcal{F}_n$ forbidden arc transportation (FAT) problems are defined for $k \in \{4, \ldots, n\}$, and $\lambda_k \in \Lambda_k(X)$ where $\Lambda_k(X)$ is the set of all possible weight vectors feasible for writing X/k as a convex combination of pedigrees in P_k. (These are explained in the illustrative examples in Section 4. For details see [4].)

3 Solving for Feasible Solutions for MI-Relaxation Given a Vertex of SEP

In this section we consider the problem of finding feasible solutions in $\mathcal{F}_n$ for a given u in SEP_n. From the MI-relaxation we know that this problem is equivalent to finding nonnegative solutions to a set of linear equations corresponding to MI-relaxation, with the slack variable values fixed at the given u. Let the given $u \in SEP_n$ be fixed throughout the discussion. We state the problem as:

Problem 3.1 *Given a $u \in SEP_n$, find $\mathcal{X} = \{X | (X, u) \in \mathcal{F}_n\}$.*

Without loss of generality we assume that u is fractional, as otherwise the problem is trivial.

We state a lemma without proof that helps us to simplify Problem 3.1.

Lemma 1 *While solving Problem 3.1, it is sufficient to impose on X nonnegativity restrictions and constraints (5).*

Applying Lemma 1, Problem 3.1 reduces to finding

$$\mathcal{X} = \{X \geq 0 | A_{[n]}X + u = \begin{pmatrix} \mathbf{1}_3 \\ \mathbf{0} \end{pmatrix}\}.$$

Let $\mathbf{u}_k = (u_{1k}, \ldots, u_{k-1k})', k = (4, \ldots, n)$. From the structure of $A_{[n]}$, it can be shown that any $\mathbf{x}_n$ satisfying

$$\begin{aligned} M_{n-1}\mathbf{x}_n &= \mathbf{u_n} & (8) \\ x_n(e) &\leq 1 - u(e) \; \forall \; e \in E_{n-1} & (9) \\ \mathbf{x}_n &\geq 0 & (10) \end{aligned}$$

can be extended to an $X = (\mathbf{x}_4, \ldots, \mathbf{x}_n)$ such that $X \in \mathcal{X}$.

The system above is a LP feasibility region with a simple structure. Now consider the capacited perfect b-matching problem, given earlier as Problem 1.1.

This problem can be solved efficiently in strongly polynomial time [2]. A practical implementation of a weighted version of the problem and its variations is given in [12].

Recall that M_{n-1} is the node-pair incidence matrix, and so we have a nice structure for the constraint matrix. Constraints (8) can be rewritten as:

$$x_n(\delta(i)) = u_{in}, \text{ for all } i \in \{1, \ldots, n-1\}.$$

It is apparent that we indeed have a capacited perfect b-matching problem here. We have the problem defined over the complete graph with $n-1$ vertices, with capcities $1 - u(e), e \in E_{n-1}$ and $b = \mathbf{u}_n$. However the solution can be fractional. So we have a relaxation of a capacited perfect b-matching problem. We can solve this relaxed problem using a maximal flow routine as done in [2]. Set up a network with a single source (0) and a single sink $(*)$ with $2(n-1)$ intermediate nodes, that is, i, i' for each $i \in \{1, \ldots, n-1\}$ respectively. For each i, arcs $(0, i)$ and $(i', *)$ both have capacity u_{in}. Arc (i, j') has capacity $1 - u_{ij}$ for $i \neq j$. The problem is to maximise the flow in the network. Let g be an optimal feasible flow in the network, then we obtain our solution $x_n(e) = 1/2[g_{ij'} + g_{ji'}]$, for $e = (i, j) \in E_{n-1}$.

However we study this problem further and obtain some results on the uniqueness of $\mathbf{x}_n$.

Lemma 2 *If $u_{in} = u_{jn} = 1$ for some i, j then in every $X \in \mathcal{X}$, $x_n(e) = 1$ for $e = (i, j)$.*

Lemma 3 *If $u_{in} = 1$ for some i then in every $X \in \mathcal{X}$, $x_n(e) = u_{ij}$ for all $e = (i, j) \in E_{n-1}$.*

Next result is a necessary condition on $\mathbf{u}_n$.

Lemma 4 *Let $\epsilon = 1 - max_i u_{in}$, and let the maximum in the definition, be attained for $i = i^*$. For any $i \in \{1, \ldots, n-1\}, i \neq i^*$, if $u_{i*i} = 1$, then $u_{in} \leq \epsilon$.*

We apply these ideas in the next section in solving for X when the given u is a vertex of SEP_n.

4 An application of FAT problems

Boyd and Pulleyblank in [9] study optimisation over SEP_n. They study the class of facet inducing inequalities for the $STSP$ polytope called *clique tree inequalities* introduced by Grötschel and Pulleyblank. They show that the vertices of SEP_n that maximise clique tree inequalities are half-integer. To show that there are vertices of SEP_n with more complicated structure, they give two classes of extreme points of SEP_n for $n = 2l + 4$, with $l \geq 3$, depending on l is odd or even. It is not known whether these vertices are also in the corresponding $STSP$ polytopes. We address this question for $l = 3, 4$ or $n = 10, 12$ respectively, and illustrate the applicability of the characterisation results involving FAT problems.

Example: 4.1 *Consider the extreme point of SEP_{10} given by:*

$u_{1,2} = u_{7,8} = u_{4,10} = u_{5,9} = 1/3$; $u_{1,3} = u_{2,3} = u_{3,4} = u_{5,6} = u_{6,7} = u_{6,8} = u_{9,10} = 2/3$; $u_{1,5} = u_{2,9} = u_{4,8} = u_{7,10} = 1$.

First we find $\{X | (X, u) \in \mathcal{F}_n\}$ using the results derived in the last section.

Consider $k = 10$. We have the problem of finding $\mathbf{x}_{10}$ feasible for the sub problem given by constraints (8) through (10). Since $u_{9,10} = 2/3, u_{4,10} = 1/3$, and $u_{7,10} = 1$. We have a fractional capacited b- matching problem involving nodes $\{4, 7, 9\}$. Capacities are all 1 and and the nodes 4, 7, and 9 have the corresponding $u_{i10} > 0$. Lemma 3 is applicable and we find the unique solution as $x_{10}(4, 7) = 1/3, x_{10}(7, 9) = 2/3$. In general we eliminate the component $\mathbf{x}_k$ from the MI-relaxation and update the slack variables using

$$\hat{u}_e = u_e + x_k(e), \ \forall e \in E_{k-1}.$$

Set $k = k - 1; u = \hat{u}$ and repeat.

Now, we have $\hat{u}_{47} = 1/3, \hat{u}_{79} = 2/3$ with $\hat{u}_e = u_e$ for the rest. So we have the new sub problem for $k = 9$ involving nodes $\{2, 5, 7\}$ with $u_{29} = 1, u_{59} = 1/3$, and $u_{79} = 2/3$. Capacities are all 1. Situation is similar and we get $x_9(2, 5) = 1/3, x_9(2, 7) = 2/3$. Proceeding in this way, it can be verified that we have a unique X, given by

$x_4(e) = 1/3, e \in E_3$; $x_5(1, 2) = 1/3, x_5(1, 4) = 2/3$; $x_6(2, 4) = 1/3, x_6(4, 5) = 2/3$; $x_7(2, 4) = x_7(2, 6) = x_7(4, 6) = 1/3$; $x_8(4, 6) = 2/3, x_8(4, 7) = 1/3$; $x_9(2, 5) = 1/3, x_9(2, 7) = 2/3$ *and* $x_{10}(4, 7) = 1/3, x_{10}(7, 9) = 2/3$.

Given this X, we consider the associated FAT_k problems.

For $k = 4$, with $\lambda_4 = \mathbf{x}_4 = (1/3, 1/3, 1/3)'$ the $FAT_4(\lambda_4)$ problem with the unique feasible flow (given along the arcs) is shown in Figure 1. Let λ_5 be the weight vector corresponding to this flow.

Similarly it is seen that the FAT problems for $k = 5, 6$, and 7 are feasible and the flows are unique as well. However for $k = 8$, $FAT_8(\lambda_8)$ is infeasible.(See Figure 2.) Hence from the uniqueness of λ_8, Theorem 5.3 in [4] asserts that $X/9 \notin conv(P_9)$. Which implies $X \notin conv(P_{10})$. And so from the uniqueness of X, we are able to conclude that $u \notin Q_{10}$, the $STSP$ polytope.

Example: 4.2 *Consider the support graph given in Figure 3 for an extreme point of SEP_{12} from Boyd and Pulleyblank's class for even l.*

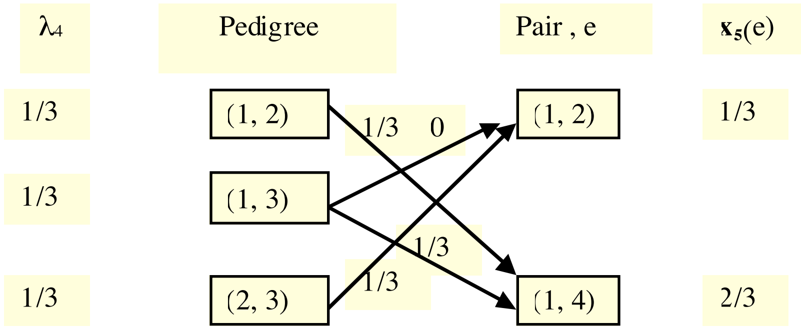

Figure 1: $FAT_4(\lambda_4)$ *Problem for* $u \in SEP_{10}$.

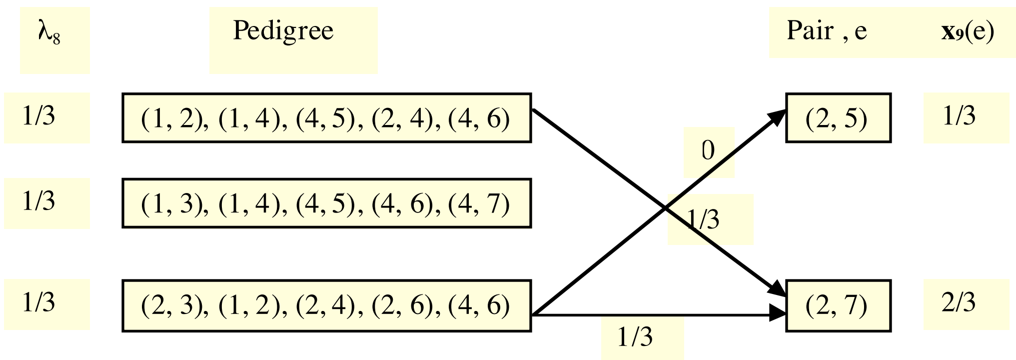

Figure 2: $FAT_8(\lambda_8)$ *Problem for* $u \in SEP_{10}$.

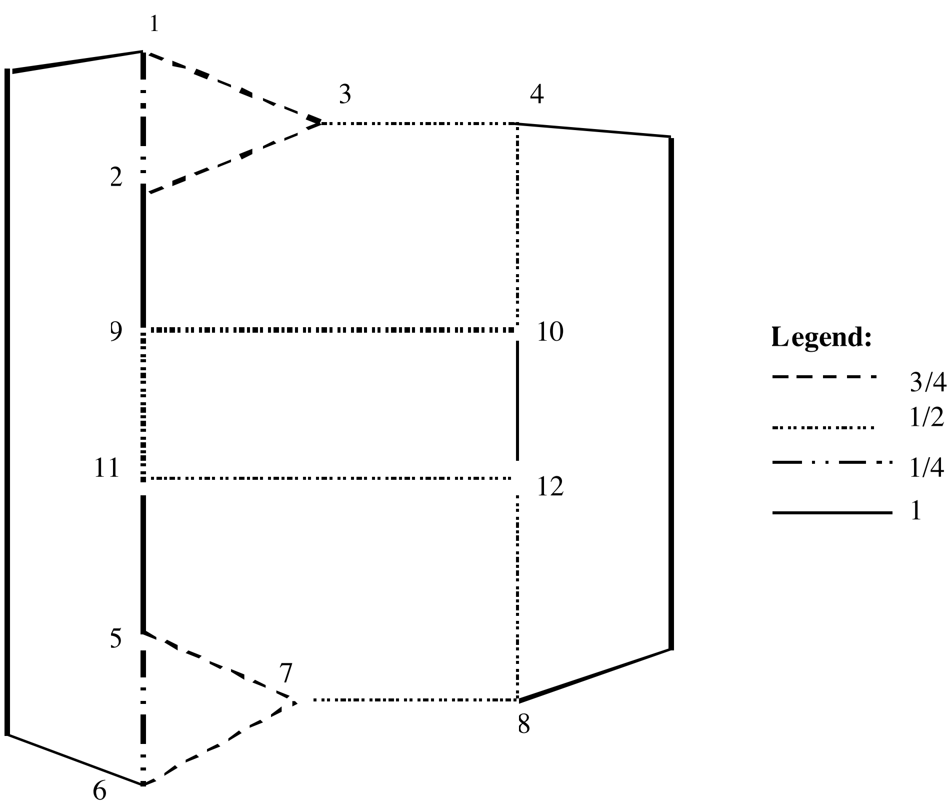

Figure 3: Support Graph of a $u \in SEP_{12}$

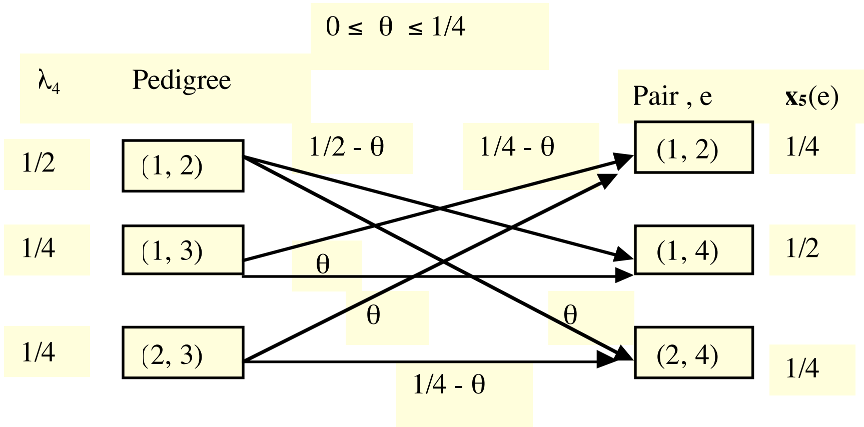

Figure 4: FAT_4 problem for given $u \in SEP_{12}$

As in the previous example we can easily see that $x_{11}(5,9) = x_{11}(5,10) = 1/2$ *and* $x_{12}(8,10) = x_{12}(10,11) = 1/2$ *are the unique solutions to the sub problems for* $k = 11, 12$ *respectively. However for* $k = 10$*, we have the new sub problem involving nodes* $\{4,5,8,9\}$ *with* $u_{4,10} = u_{5,10} = u_{8,10} = u_{9,10} = 1/2$ *with capacities of all edges being* 1 *but for* $(4,8)$*. Since* $u_{48} = 1$*, we have the corresponding capacity zero. We find two extreme point solutions for the sub problem for* $k = 10$*. Proceeding further we have two extreme points* X, Y *corresponding to the given* u*.*

X *is given by*

$x_4(1,2) = 1/2, x_4(1,3) = x_4(2,3) = 1/4$; $x_5(1,2) = x_5(2,4) = 1/4, x_5(1,4) = 1/2$; $x_6(1,4) = 1/4, x_6(1,5) = 3/4$; $x_7(4,5) = x_7(4,6) = 1/4, x_7(5,6) = 1/2$; $x_8(2,4) = x_8(4,7) = 1/2$; $x_9(2,5) = x_9(2,8) = 1/2$; $x_{10}(4,5) = x_{10}(8,9) = 1/2$; $x_{11}(5,9) = x_{11}(5,10) = 1/2$ *and* $x_{12}(8,10) = x_{12}(10,11) = 1/2$.

and Y *is given by*

$Y/7 = X/7$; $y_8(4,5) = y_8(4,7) = 1/2$; $y_9(2,5) = y_9(2,4) = 1/2$; $y_{10}(4,9) = y_{10}(5,8) = 1/2$; $\mathbf{y}_{11} = \mathbf{x}_{11}$; $\mathbf{y}_{12} = \mathbf{x}_{12}$.

Any convex combination, $X^\mu = \mu X + (1-\mu)Y, 0 \leq \mu \leq 1$ *of these two extreme points is feasible for the given* u*. Therefore* $X^\mu = (X/7; \mu\mathbf{x}_8 + (1-\mu)\mathbf{y}_8; \ldots; \mu\mathbf{x}_{10} + (1-\mu)\mathbf{y}_{10}; \mathbf{x}_{11}; \mathbf{x}_{12}), 0 \leq \mu \leq 1$*. So if we are able to show that a* FAT_6 *problem has a unique feasible flow it applies to all* X^μ*.*

Consider $FAT_4(\lambda_4)$ *as shown in in Figure 4. We have a feasible flow* f_θ*, for each* $0 \leq \theta \leq 1/4$*. However,* $FAT_5(\lambda)$ *problem, is feasibile only if* $\theta = 1/4$*. Obviously this leaves us with a unique feasible flow yielding* λ_6*. It can be checked that* $FAT_6(\lambda_6)$ *problem also has a unique feasible flow yielding* λ_7 *which puts equal weight* $(1/4)$ *on the following pedigrees:*

$[(1,2),(1,4),(1,5),(5,6)], [(1,2),(2,4),(1,4),(4,6)]$, $[(1,3),(1,4),(1,5),(4,5)]$ *and* $[(2,3),(1,2),(1,5),(5,6)]$.

For $k = 7$*, we have* $FAT_7(\lambda_7)$ *problem with respect to* $\mu\mathbf{x}_8 + (1-\mu)\mathbf{y}_8$ *with* $\mu \in [0,1]$*.* $FAT_7(\lambda_7)$ *is feasible with a unique flow as shown in Figure 5 for any* $\mu \in [1/2, 1]$*. In the next step we find* $FAT_8(\lambda_8)$ *is infeasible as* λ_8 *puts a weight of* $1/4$ *on the pedigree* $[(1,3),(1,4),(1,5),(4,5),(4,7)]$*. And all the arcs leaving this origion are forbidden, for any choice of* μ*. So* $X^\mu/9 \notin conv(P_9)$ *for any* μ *. Hence* X^μ *is not in* $conv(P_{12})$ *for any* μ*. Or we have shown* $u \notin Q_{12}$*.*

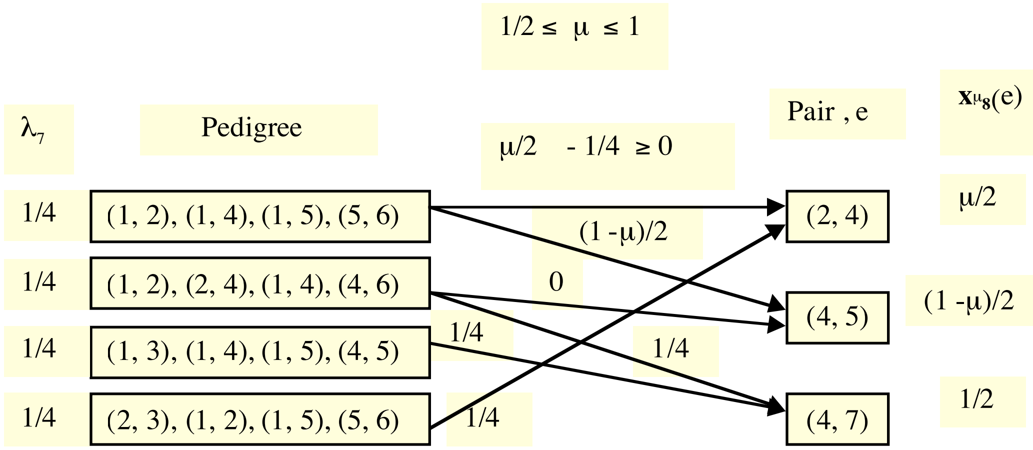

Figure 5: FAT_7 problem for given $u \in SEP_{12}$

These examples are given to illustrate how FAT problems can be used to check membership in Q_n. Similar steps in general may not be tractable. However an algorithm needs to be developed to solve problems of this kind in general. Current research is directed towards this. Recently the author has shown that given two pedigrees, testing whether they are non adjacent vertices in $conv(P_n)$ can be done in polynomial time [5]. This result is encouraging as it is known that such a problem is NP-complete, with respect to tours in Q_n [13].

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