Some Initial Ideas Regarding a Replenishment Decision Involving Partial Postponement

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Abstract

In this paper we consider the choice of the order quantities for a group of end items, each facing random demand in a period of interest. There is the additional feature of having W units of unfinished stock that can be customized to produce any of the end items once demands materialize. The paper was motivated by a practical situation in the fast food industry. Some interesting analytical results are developed as well as suggestions for dealing with the case where there are a large number of distinct items.

1 Introduction

The development of this paper was motivated by the problem encountered by a student team in an applied project in a masters level course. Specifically, an entrepreneur had recently launched a pizza operation with very specific characteristics. In particular, all pizzas offered are of the same size, thus simplifying preparation and packaging requirements. Moreover, most of the production is carried out centrally with part of the sales being through satellite locations, such as in shopping centres, where local baking of the products is carried out. Of particular interest is the possibility of having some limited finishing capability at a satellite location, i.e. if sufficient unfinished shells and toppings are available, up to W pizzas can be finished per day. Hence, if one runs out of finished pizzas of a particular type, additional demand can still be met by appropriately adding toppings to unfinished pizzas. Leftover finished pizzas are disposed at a small unit salvage value whereas leftover unfinished pizzas and toppings can be carried in inventory if properly refrigerated. Thus, for any given day the decisions for a given satellite would be the quantities of each type of finished pizza to have on hand plus the quantity of unfinished pizzas and associated toppings.

Holding a limited stock of unfinished pizzas is tantamount to a partial postponement strategy. Part of the stock is retained in a partially processed stage and customized once demands are realized. In addition to the replenishment quantity decisions a more strategic issue is the selection of the finishing capacity W. Swaminathan and Tayur [7] have looked at a more general version of this problem where components are assembled into finished products and there is limited assembly capability after the demands are known.

In Section 2 we more precisely specify the sequence of events each day, the decision variables involved and the assumption regarding the demand distributions. Section 3 briefly discusses two possible objectives. Most of the rest of the paper focusses on developing some results and other insights for the case of independent, normally distributed, demands for the different end items (types of pizza). In particular, in Section 4 we discuss the distribution of the requirements for the unfinished units of stock as a function of the replenishment quantities of each of the finished items. If there is a large variety of end items (pizza types), then the distribution should tend to be normally distributed which, as shown in Section 5, permits the representation of the expected profit as a function of the replenishment quantities, the finishing capacity, the means and standard deviations of the demands of the different types of end items (pizzas), and certain economic parameters. Unfortunately, there does not appear to be an analytic method (such as by setting the first derivatives to zero) for finding the best values of the decision variables. This is a form of stochastic decision problem with recourse (one commits to stock of finished and unfinished items, then once the random demands are known, one uses available stocks in the best possible fashion). This type of problem is inherently difficult to solve mathematically (see, for example, Swaminathan and Tayur [7], Jönsson, Jörnsten and Silver [3], Rockafeller and Wets [5]). In Section 6 special insights are obtained for the case of but a single type of pizza, specifically this permits obtaining lower and upper bounds on the replenishment size of that type of pizza in the more general case of multiple types of end items. Some numerical illustrations are presented in Section 7. Next, in Section 8 we suggest possible search procedures for finding appropriate values of the decision variables when there are a large number of end items. Section 9 provides some summary comments including mention of a very different, potential solution procedure. Some of the technical details are presented in the appendix.

2 A More Detailed Specification of the Problem

Consider the sequence of events within a single day at a particular satellite location. One receives a quantity Q_i of finished pizza type i ($i = 1, 2, ..., n$) where n is the number of types of pizzas. Moreover, stocks of unfinished pizzas and associated toppings are raised so as to be able to produce W finished pizzas (the finishing capacity at the satellite location). This latter, order-up-to-type, policy is appropriate because of the ability to carry forward unused, unfinished pizzas and toppings. Next, the demands for the various types of pizzas occur during the day. These demands are first satisfied from the stock of finished items. Then, once one or more items have their stocks depleted, demands for these items are satisfied on a first-come-first-serve basis by applying appropriate toppings to unfinished pizzas. At the end of the day any remaining stocks of finished items are salvaged.

The decision variables are the Q_i's (W is of a more strategic nature in that increasing W would require more equipment and/or labour at the satellite location). It is seen that the Q_i decisions from day to day are separable.

As mentioned earlier, we deal with the situation where the demands for the different types of end items can be represented as independent normally distributed variables (where parameters could certainly be varied by day of the week or time of the year). Limited data from the organization showed that the normal assumption was reasonable, however, independence may be somewhat questionable in that a short term, unanticipated event, such as inclement weather, might affect the demand levels for all types of pizzas. For a particular day we denote by μ_i and σ_i the mean and standard deviation of the demand for finished item i.

3 Possible Objectives

There are at least two plausible objectives to be used in selecting the values of the decision variables namely i). maximizing the expected total profit in the day, or ii). minimizing expected total relevant costs per day subject to providing a certain customer service level (e.g. on average, satisfying at least 95% of the demands). In this paper we focus on the former.

4 The Distribution of the Requirements for the Unfinished Pizzas

We first introduce some further notation: Let x_i be the (normally distributed with mean μ_i and standard deviation σ_i) demand for finished item i $(i = 1, 2, ..., n)$, and Q_i be the replenishment quantity for finished item i $(i = 1, 2, ..., n)$ with

$$Q_i = \mu_i + k_i \sigma_i \tag{1}$$

Also let y_i be the demand for unfinished pizzas resulting from demand for item i not being satisfied by Q_i $(i = 1, 2, ..., n)$ and $Y = \sum_{i=1}^{n} y_i$ be the total demand for unfinished pizzas. Then,

$$y_i = \max(0, x_i - Q_i). \tag{2}$$

The distribution of y_i will be a spike at 0 with probability mass $\Phi(k_i)$, i.e. the probability that a unit normal variable takes on a value less than k_i, and a continuous distribution, which is the tail of a normal distribution, for positive values of y_i. One can show (see, for example, Silver, Pyke and Peterson [6]) that the expected value of y_i is

$$E(y_i) = \sigma_i G_u(k_i) \tag{3}$$

$$\text{where } G_u(k_i) = \int_{k_i}^{\infty} (u - k_i)\phi(u)du = \phi(k_i) - k_i \left[1 - \Phi(k_i)\right] \tag{4}$$

is the so called normal loss function with $\phi(z)$ being the unit normal probability density function. In Part 1 of the appendix we show that

$$Var(y_i) = \sigma_i^2 \left\{ J_u(k_i) - [G_u(k_i)]^2 \right\} \tag{5}$$

$$\text{where } J_u(k_i) = \int_{k_i}^{\infty} (u - k_i)^2 \phi(u)du = (1 + k_i^2) \left[1 - \Phi(k_i)\right] - k_i \phi(k_i) \tag{6}$$

The $\Phi(k_i)$, $G_u(k_i)$ and $J_u(k_i)$ functions can be evaluated using spreadsheets or tables as described in [6].

Unfortunately for small n there is no simple representation of the probability distribution of Y. To illustrate, for the case of $n = 2$, Y will have a spike at zero with mass $\Phi(k_1) \cdot \Phi(k_2)$ but any non-zero value of Y can occur in three distinct ways: i). the demand for item 1 is less than Q_1 and the demand for item 2 is $Q_2 + Y$, ii). the demand for item 1 is $Q_1 + Y$ and the demand for item 2 is less than Q_2, and iii). both items have unsatisfied demands with the total being Y (this involves a convolution of two random variables). However, as n becomes reasonably large one would expect that Y tends to normality with

$$\mu_Y = \sum_{i=1}^{n} E(y_i) = \sum_{i=1}^{n} \sigma_i G_u(k_i) \tag{7}$$

$$\sigma_Y^2 = \sum_{i=1}^{n} Var(y_i) = \sum_{i=1}^{n} \sigma_i^2 \left\{ J_u(k_i) - [G_u(k_i)]^2 \right\} \tag{8}$$

We make use of this limiting distribution in the next section.

5 Expected Total Profit

We introduce the following economic parameters:

- v_i unit cost of a finished item (pizza type) i $(i = 1, 2, ..., n)$
- p_i selling price of finished item i $(i = 1, 2, ..., n)$
- g_i salvage value per unit of item i $(i = 1, 2, ..., n)$
- $\bar{p}$ average selling price of items that are finished to order
- $\bar{v}$ average unit value of an unfinished pizza and associated topping

One could also include a penalty (beyond the lost profit) for each unit of demand not satisfied. Note that $\bar{p}$ and $\bar{v}$ will, in general, depend upon the Q_i values, hence can not be prespecified.

The expected total profit, $ETP(k_i\text{'s}, W)$, is composed of five components, viz.

$$ETP(k_i\text{'s}, W) = -\sum_{i=1}^{n} Q_i v_i + \sum_{i=1}^{n} ER(Q_i) + \sum_{i=1}^{n} ES(Q_i) + ER(W) - ERC(W) \tag{9}$$

where $ER(Q_i)$ is the expected revenue from direct sales from the Q_i units, $ES(Q_i)$ is the expected value of units salvaged (left over) from the Q_i units, $ER(W)$ is the expected revenue from sales from the W unfinished units, and $ERC(W)$ is the expected replacement cost of the units of unfinished pizzas and toppings sold. Using a basic newsvendor formulation (see Silver, Pyke and Peterson [6]), together with equation (1), leads directly to

$$ER(Q_i) = p_i \left[\mu_i - \sigma_i G_u(k_i) \right] \tag{10}$$

and

$$ES(Q_i) = g_i \sigma_i \left[k_i + G_u(k_i) \right] \tag{11}$$

Moreover, with Y being normally distributed and with an available stock of W unfinished units, there is a direct analogy with (10), namely

$$ER(W) = \bar{p} \left[\mu_Y - \sigma_Y G_u \left(\frac{W - \mu_Y}{\sigma_Y} \right) \right] \tag{12}$$

Also, the number of units of unfinished stock to be replenished equals the number of units sold. Hence

$$ERC(W) = \frac{\bar{v}}{\bar{p}} ER(W) \tag{13}$$

$$\text{or} \qquad ER(W) - ERC(W) = \left(1 - \frac{\bar{v}}{\bar{p}}\right) ER(W) \tag{14}$$

Thus, (1),(7),(8),(10),(11),(12) and (14) can be used in (9) to express the expected total profit per day as a function of the decision variables, the k_i's, as well as the μ_i's, σ_i's, W and the various economic parameters. Unfortunately, there is a complicated interaction of the k_i's through the σ_Y terms in (12) in that (8) shows that in taking the square root of the variance the effects of the k_i's are no longer separable. Thus, an attempt to set the partial derivatives (with respect to the k_i's) equal to zero, led to a very complicated set of non-linear equations. Moreover, there is no guarantee that their solution would even lead to profit maximization as we have not been able to show that ETP is a concave function of the k_i's. Thus it appears that some type of search procedure is needed to find the best (or at least a very good) solution.

Analysis of the special case of a single type of finished item results in lower and upper bounds on the best value of k_i. This is the topic of the next section.

6 Special Case of a Single Type of Finished Item (Pizza)

When there is just a single type of item we drop the subscript i. Moreover, we will be able to use the true distribution of y, the demand not met from the Q finished units (i.e. not have to assume that it is normally distributed).

There is a close analogy with the standard newsvendor problem. The demand x occurs and as much of it as possible is satisfied from the Q units of finished stock. If necessary, additional demand beyond Q is satisfied to the extent possible from the W units of unfinished stock. As shown in Part 2 of the appendix the expected profit is given by

$$\begin{aligned} EP(k,W) &= -Qv + p\,[\mu - \sigma G_u(k)] + g\sigma\,[k + G_u(k)] \\ &\quad +(p - v_0)\,[G_u(k) - G_u(k + W/\sigma)] \end{aligned} \tag{15}$$

where v_0 is the unit cost of materials and preparation of a pizza from an unfinished pizza plus topping. (We have used p and v_0, instead of $\bar{p}$ and $\bar{v}$, because for the case of a single type of finished item there is no uncertainty about the unit revenue and unit cost of a pizza that is finished at the satellite location). In general, we would expect $v_0 > v$. In Part 2 of the appendix it is shown that $EP(k,W)$ is a concave function of k. Thus, setting $dEP(k,W)/dk = 0$ will produce the maximum expected profit and the result is that the associated k^L must satisfy

$$\Phi(k^L) + \frac{p - v_0}{v_0 - g}\Phi(k^L + W/\sigma) = \frac{p - v}{v_0 - g} \tag{16}$$

The k^L from (16) provides a lower bound (hence the symbol L) on the associated k for the multi-item problem in that (16) was derived based on the assumption that all of W is available for the item under question. Also, by setting $W = 0$ in (16) we are assuming that none of the unfinished stock is available for the specific item, hence we obtain an upper bound k^U on the associated k for the multi-item problem. Setting $W = 0$ in (16) leads to

$$\Phi(k^U) = \frac{p - v}{p - g} \tag{17}$$

which, as expected, is the solution to the standard newsvendor problem under normally distributed demand. Note that (17) indicates that k^U is independent of σ, but (16) shows that k^L depends on σ.

7 Some Numerical Illustrations

In this section we examine two examples with small numbers of end items which permit an exhaustive search over the feasible ranges of the Q_i's (the lower and the upper bounds, Q_i^L and Q_i^U, are found through the aforementioned lower and upper bounds on the k_i's). For each set of Q_i's simulation was used to estimate the expected profit (as it was felt inappropriate to use the normality assumption for Y for such a small number of items).

The first example involves two items that are identical except the demand for item 1 is considerably more variable than for item 2. The solutions (Q_i^*'s) are obtained for two different values of the finishing capacity W. The parameter values and solutions are shown in Table 1.

Item Characteristics

i	μ_i	σ_i	p_i	v_i	g_i	
1	40	12	10	5	2	$\bar{p} = 10$
2	40	2	10	5	2	$\bar{v} = 6$

Solution for $W = 0$

i	Q_i^* (based on k_i^U)	
1	44	$ETP^* = 357.42$
2	41	

Solution for $W = 6$

i	Q_i^L	Q_i^U	Q_i^*	
1	40	44	41	$ETP^* = 367.59$
2	38	41	40	

Solution for $W = 12$

i	Q_i^L	Q_i^U	Q_i^*	
1	38	44	38	$ETP^* = 373.48$
2	38	41	40	

Table 1: Example 1

The maximum expected total profits are 357.42, 367.59 and 373.48 for $W = 0$, 6 and 12, respectively. As expected, there are diminishing returns from increasing W. Moreover, if one implemented the solution based on assuming $W = 0$ when, in fact W was 6 or 12, the resulting expected total profits would be 365.55 and 368.87, respectively.

Of particular interest is the behaviour of Q_1^* and Q_2^* as W is increased. When $W = 0$, both items have positive k_i's. With $W = 6$, we still have $k_1^* > 0$ but $k_2^* = 0$. Then, for $W = 12$, $k_1^* < 0$ and k_2^* stays at 0. There is an interactive effect between k_i^* and σ_i. This is seen analytically through (16) where the lower bound k value is seen to depend on σ. In fact, the k^L of (16) decreases as W/σ increases, i.e. the more unfinished capacity (W) we have (dedicated to that item) relative to the variability of the demand of the item, the less finished stock should be ordered for the item.

A second example was constructed with the same two items as in the first example but with the addition of a third item with an intermediate value of variability, $\sigma_3 = 6$. For the case of $W = 12$ we obtained the solution $Q_1^* = 40$, $Q_2^* = 40$, $Q_3^* = 39$. This result is non-intuitive in that the only item with a negative safety factor is item 3 which has the intermediate value of σ. Again, this is due to the interactive effect between k_i^* and σ_i. Note that compared to the $W = 12$ case of example 1, Q_1^* has increased and Q_2^* has stayed the same. The increase in Q_1^* is due to the need to now share W with a third item.

8 A Possible Approach to Maximize Expected Total Profit

When there are a large number of items exhaustive search of all possible Q_i combinations (bounded by Q_i^L and Q_i^U) becomes impossible. All the, admittedly small, examples examined so far appear to have a concave objective function. If the function is indeed concave for larger problems, then one could use some form of steepest ascent routine to achieve the optimum. If not, then to avoid converging to a local optimum use of a more elaborate metaheuristic would be more appropriate. Possibilities include tabu search (Glover [2]), simulated annealing (Eglese [1]) and genetic algorithms (Mitchell [4]). For a large number of items the normal approximation of the distribution of Y is likely to be quite accurate, thus the equations of Section 5 can be used to compute the ETP for any candidate set of Q_i's. Moreover, although the partial derivative of ETP with respect to k_i is complicated, it could be calculated and used as part of a gradient search.

9 Summary

In this paper we have examined a replenishment problem involving partial postponement, specifically a limited stock of unfinished units is available to satisfy demands once the stocks of one or more finished items are depleted. A number of insights have been developed for dealing with this problem, but a number of extensions can be pursued.

First, the performance of different search procedures has to be investigated. Second, even if one or more perform well, there would still be considerable value in having a heuristic procedure that quickly provides ballpark estimates of the k_i's. In addition, some of the assumptions could be relaxed, in particular, the independence of the demands for the different end items. One possibility would be to use a limited set of scenarios of demands similar to the approach by Swaminathan and Tayur [7].

Appendix - Derivations

1 Variance of y_i

For simplicity we suppress the i subscript.

$$E(y^2) = \int_Q^\infty (x - Q)^2 f(x) dx$$

$$= \int_{\mu+k\sigma}^{\infty} [x - (\mu + k\sigma)]^2 \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Set $u = (x - \mu)/\sigma$ to obtain

$$E(y^2) = \sigma^2 \int_k^{\infty} (u - k)^2 \phi(u) du = \sigma^2 J_u(k) \quad \text{(A.1)}$$

Then, $Var(y) = E(y^2) - [E(y)]^2$. Substituting from equations (3) and (A.1)

$$Var(y) = \sigma^2 \left\{ J_u(k) - [G_u(k)]^2 \right\}$$

2 The Case of a Single Type of End Item (Pizza)

The expected profit has the same general terms as in equation (9) and the $ER(Q)$ and $ES(Q)$ are as in (10) and (11). Hence,

$$EP(k, W) = -Qv + p[\mu - \sigma G_u(k)] + g\sigma [k + G_u(k)] + (p - v_0)EUS(W) \quad \text{(A.2)}$$

where we need to develop an expression for $EUS(W)$, the expected number of units sold from unfinished pizzas.

As discussed in Section 4 the excess (beyond Q) demand (y) for the end item has a spike at zero with probability $\Phi(k)$ and the p.d.f. for non-zero values of y is

$$f_y(y_0) = f_x(Q + y_0) \qquad 0 < y_0.$$

When y is 0 there are no sales from W units. Hence

$$EUS(W) = \int_Q^{Q+W} (x_0 - Q) f_x(x_0) dx_0 + W \int_{Q+W}^{\infty} f_x(x_0) dx_0$$

Using the fact that x has a normal distribution (μ,σ) and that $Q = \mu + k\sigma$, this reduces to

$$EUS(W) = \sigma [G_u(k) - G_u(k + W/\sigma)] \quad \text{(A.3)}$$

Substituting into (A.2) we obtain

$$\begin{aligned} EP(k, W) &= -\mu v - k\sigma v + p[\mu - \sigma G_u(k)] + g\sigma [k + G_u(k)] \\ &\quad + (p - v_0)\sigma [G_u(k) - G_u(k + W/\sigma)] \end{aligned}$$

$$\text{Now} \qquad \begin{aligned} \frac{dEP(k, W)}{dk} &= -\sigma v + \sigma p [1 - \Phi(k)] + g\sigma\Phi(k) \\ &\quad + (p - v_0)\sigma [\Phi(k) - 1 - \Phi(k + W/\sigma) + 1] \end{aligned} \quad \text{(A.4)}$$

where we have used $dG_u(k)/dk = \Phi(k) - 1$.

$$\begin{aligned} \frac{d^2EP(k, W)}{dk^2} &= -\sigma p\phi(k) + g\sigma\phi(k) + (p - v_0)\sigma [\phi(k) - \phi(k + W/\sigma)] \\ &= -\sigma(v_0 - g)\phi(k) - \sigma(p - v_0)\phi(k + W/\sigma) < 0 \end{aligned}$$

Hence, $EP(k, W)$ is a concave function of k and thus setting $dEP(k, W)/dk = 0$ will produce the maximum expected profit. Using (A.4) this leads to the optimal k, denoted by k^L, satisfying

$$\Phi(k^L) + \frac{p - v_0}{v_0 - g}\Phi(k^L + W/\sigma) = \frac{p - v}{v_0 - g}$$

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