Capacity Expansion Optimisation in Transportation Networks

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Abstract

The network design problem determines the optimal links in a transportation network to upgrade in order to minimise traffic congestion. Traditional approaches to this problem under an optimisation framework involve solving a bi-level, non-linear, integer program. We explore an alternative approach known as the system optimal network design problem where road users are assumed to act cooperatively. This assumption simplifies the formulation into a single-level non-linear integer program, which we linearise and solve through Kelleys cutting plane method. The effect of increasing the number of network loading conditions considered in the model is explored, and we show the importance of modelling these scenarios in unison to find the optimal solution. The structure of the network design problem with multiple scenarios lends itself to decomposition methods. We compare the solve-time performance between using Benders decomposition and a large mixed-integer program. Results show that the decomposition has greater performance as the network size and number of states increases, but suffers when the number of upgrade decisions increases.

Key words: transport, capacity expansion, system optimal, network design, decomposition.

1 Introduction

1.1 Motivation

Transportation planning is a critical task to ensure that the transport infrastructure of urban centers scales efficiently as the population of a region grows. Committing to upgrades or constructing new assets within the network are extremely costly decisions, which are paid for by public funds. Determining where and how to stage these upgrades over time in the most cost effective manner is a crucial problem to address.

Design choices made by planners alter the underlying traffic network and affect the route choices made by road users until new equilibrium flows are reached

(Mathew and Sharma 2009). Current planning practices typically involve modelling the different major network changes and performing cost benefit analysis on each change to select the best solution. These designs are restricted by economic and social factors such as a limited annual budget, or local zoning regulations governing possible network alterations. Planning through this method of trialling different solutions relies heavily on the planners' intuition and has no guarantee of finding the globally optimal set of network upgrades; this motivates the need for an optimisation approach to solving the network design problem.

1.2 Travel cost functions

The choice of route for any traveller in a transportation network may depend on any combination of several factors such as travel time, safety, or road tolls. These factors may be aggregated into a single *generalised cost* for travelling on a particular link.

Generalised cost functions are assumed throughout literature to be monotonically increasing, and therefore convex, with respect to the volume of vehicle flow on a link (Sheffi 1984). This implies that the disutility of using a link increases with the total number of users on the link. For this report, we assume without loss of generality that the travel time is the only factor in determining the cost of a route. We furthermore assume that travel times on each link are independent of all other links (Patriksson 1994).

We utilise the empirical travel time function developed by the Bureau of Public Records (BPR) (Bureau of Public Roads 1964), which relates the travel time on a link with the volume of traffic on a particular link, and follows the follows the general form:

$$t(f) = t^0 \left[1 + \alpha \left(\frac{f}{c} \right)^{\beta} \right] \tag{1}$$

where t is the travel time on a particular link with traffic flow f. t^0 denotes the *free-flow travel time* when a single user is travelling on that link; α and β are parameters calibrated from empirical data; and c is the practical capacity of a link which represents the maximum steady state flow on a link.

1.3 Traffic assignment

A common assumption made in transportation planning is that users will modify their choice of route in order to minimise their individual travel times. This results

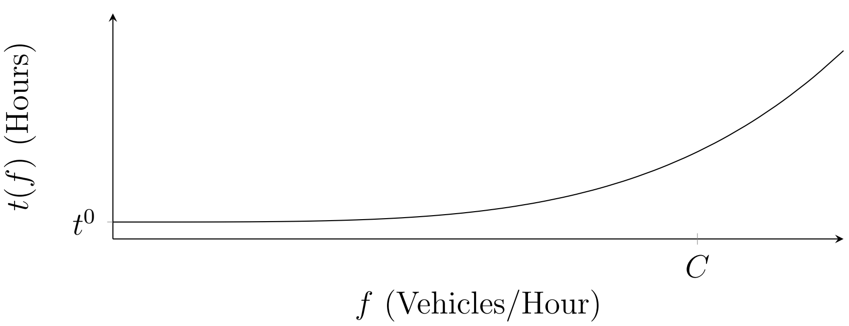


Figure 1: An example of the BPR link performance (travel time) function.

in the Wardrop equilibrium, which states that "The journey times on all routes actually used are equal, and less than those which would be experienced by a single driver on any unused route" (Wardrop 1952).

Traditional network design paradigms focus on the *user equilibrium network design problem* (UENDP), where changes to the network are made to reduce the total system-wide congestion, while road users focus on reducing their indivudal travel times. The UENDP is constructed as a bilevel optimisation problem where the upper level decisions represent network design decisions, and the lower level problem is the user equilibrium traffic assignment problem. Bilevel programs are generally non-convex optimisation problems, even in the absence of integer variables. This poses problems when attempting to solve the UENDP within an integer programming framework using branch and bound.

This project instead takes an alternative approach which focusses on a *system optimal network design problem* (SONDP), where the network changes are made assuming road users behave cooperatively to minimise the overall congestion throughout the network. When link upgrades are continuous, the SONDP is a convex non-linear optimisation program. This allows the SONDP to be solved as an integer program more easily, since branch and bound is able to be used. The design solution obtained from the SONDP provides the choice of upgrades which best reduce the potential congestion in a network.

2 System Optimal Network Design

Let a network be represented by a directed graph, $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ which includes a set of consecutively numbered nodes $\mathcal{N}$ and a set of consecutively numbered links $\mathcal{A}$. Let $\mathcal{W}_n$ denote set of set of links originating from node $n \in \mathcal{N}$, and $\mathcal{V}_n$ the set of links terminating at n.

Let $\mathcal{O}$ denote the set of nodes where users originate, and x_{ij}^o the flow of vehicles on link $(i, j) \in \mathcal{A}$ originating from $o \in \mathcal{O}$. The point to point traffic demand of users from each origin to each node is given by d_n^o, $n \in \mathcal{N}, o \in \mathcal{O}$. f_{ij} is defined to be the total flow of vehicles each link $(i, j) \in \mathcal{A}$, with corresponding travel time t_{ij}.

Define c_{ij} denote the current capacity of each link $(i, j) \in \mathcal{A}$, each with a base vehicle capacity C_{ij}. Let $\mathcal{A}^*$ denote the set of upgradable links, and z_{ij}=1 if a link $(i, j) \in \mathcal{A}^*$ is upgraded, and 0 otherwise. Let $\bar{c}_{ij}$ denote the increase in capacity if a link $(i, j) \in \mathcal{A}^*$ is upgraded with cost b_{ij} under a capital budget limit of B.

The system optimal network design problem can be formulated as:

$$\min \quad \sum_{(i,j)\in\mathcal{A}} t_{ij}(f_{ij}, c_{ij}) f_{ij} \tag{2}$$

$$\text{s.t.} \quad \sum_{(i,j)\in\mathcal{A}^*} b_{ij} z_{ij} \leq B \tag{3}$$

$$c_{ij} = C_{ij} + \bar{c}_{ij} z_{ij} \qquad \forall (i,j) \in \mathcal{A} \tag{4}$$

$$\sum_{(n,j)\in\mathcal{W}_n} x_{nj}^o - \sum_{(j,n)\in\mathcal{V}_n} x_{jn}^o = d_n^o \qquad \forall n \in \mathcal{N}, o \in \mathcal{O} \tag{5}$$

$$x_{ij}^o \geq 0 \qquad \forall (i,j) \in \mathcal{A}, o \in \mathcal{O} \tag{6}$$

$$f_{ij} = \sum_{o\in\mathcal{O}} x_{ij}^o \qquad \forall (i,j) \in \mathcal{A} \tag{7}$$

The objective function (2) minimises the total travel time experienced over all road users in the system, and assumes that road users behave cooperatively. Constraint (3) specifies the budget limit for upgrades, and constraint (4) relates upgrade decisions z_{ij} to the capacity of each link. Constraints (5 -7) ensure that point to point traffic demands are met.

2.1 Kelleys cutting planes

The SONDP is a linearly constrained non-linear optimisation problem. The contribution of each link $(i, j) \in \mathcal{A}$ to the objective function is given by:

$$\Theta = t_{ij}(f_{ij}, c_{ij})f_{ij} = t_{ij}^0 \left[f_{ij} + \alpha_{ij} \frac{f_{ij}^{\beta_{ij}+1}}{c_{ij}^{\beta_{ij}}} \right] \tag{8}$$

whose Hessian is positive semidefinite for all nonnegative values of f and c (given valid parameter values for α, β, C and $\bar{c}$). Because the objective function is convex, any supporting hyperplane provides global bounds on the objective value of the problem.

Kelley's cutting plane method begins with a course approximation to the objective, which is then iteratively refined until a specified tolerance is met. These cuts take the form:

$$\begin{aligned} \theta_{ij} \geq\ & \frac{\partial \Theta_{ij}(\hat{f}_{ij,k}, \hat{c}_{ij,k})}{\partial f_{ij}} (f_{ij} - \hat{f}_{ij,k}) \\ & + \frac{\partial \Theta_{ij}(\hat{f}_{ij,k}, \hat{c}_{ij,k})}{\partial c_{ij}} (c_{ij} - \hat{c}_{ij}) \\ & + \Theta_{ij}(\hat{f}_{ij,k}, \hat{c}_{ij,k}) \qquad \forall (i,j) \in \mathcal{A}, k \in \{1, \ldots, K\} \end{aligned} \tag{9}$$

where $\hat{f}_{ij,k}$ and $\hat{c}_{ij,k}$ are the discrete flows and capacities where cutting planes are applied, and θ_{ij} is now a surrogate variable for the non-linear objective function terms. Figure 2 shows a close-up of the objective function on a given link in the network over the domain $f = [1400, 1600], c = [800, 1000]$. A cutting plane generated at $(\hat{f}, \hat{c})$=(1500,900) forms a first order approximation to the objective function.

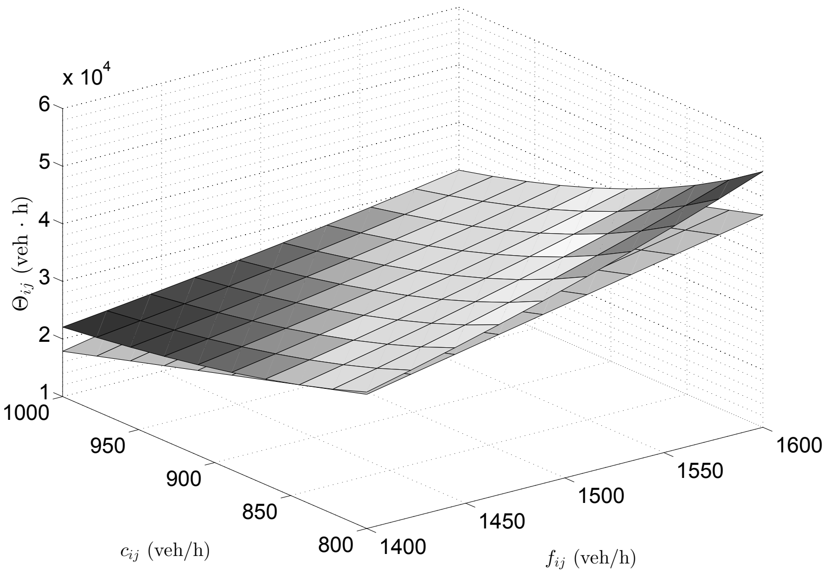


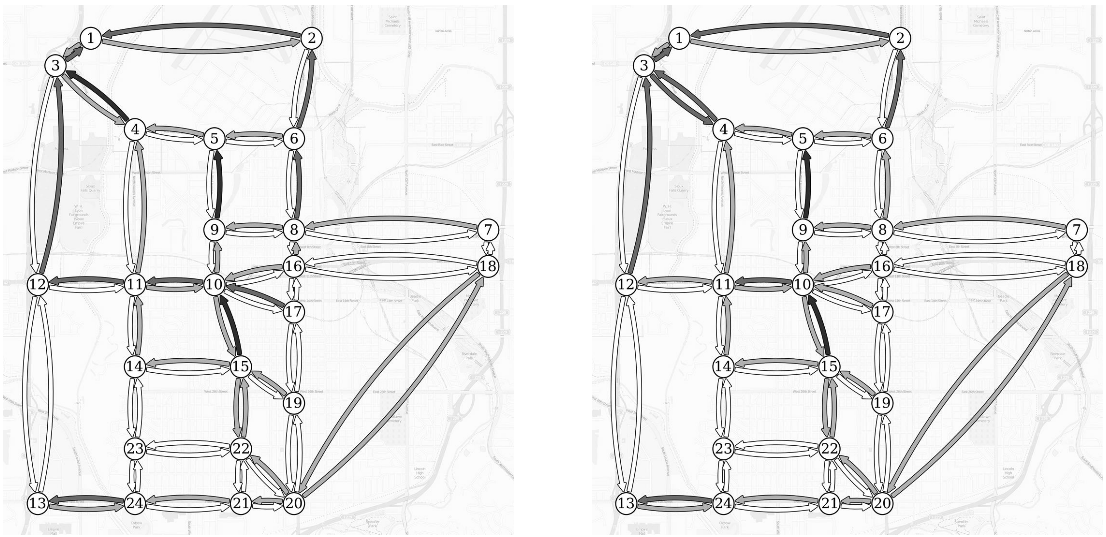
Figure 2: A Kelley's cutting plane forming a linear approximation on the SONDP objective function.

2.2 Visualisation of the SONDP

Network data used for the computational results in this paper were taken from a repository of test problems (Bar-Gera 2013). After verifying the integrity of the developed models, some data sets were modified in order to further explore aspects of the model. In particular, the traffic patterns in this section differ from those from the original data.

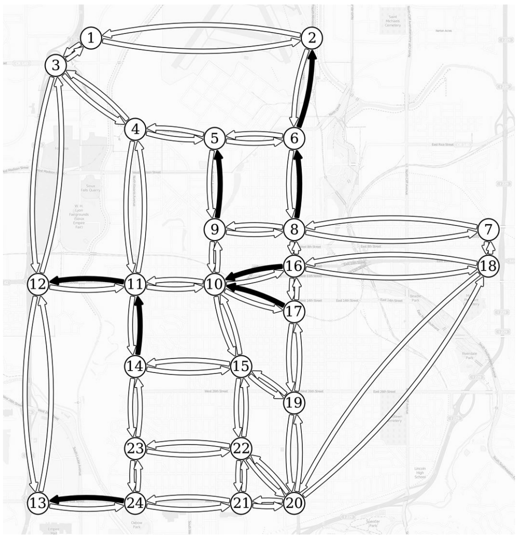
Figure 3a shows the original congestion in a modified version of the Sioux Falls transportation network; Figure 3b shows the congestion after upgrades are made; and Figure 3c shows the optimal links to upgrade.

It can be seen that the optimal policy for upgrading links does not necessarily correpond to upgrading the most congested links. For instance, link (4,3) is heavily congested but its capacity is not expanded. This can be attributed to the fact that there are two methods of reducing travel times: directly upgrading links, and diverting flow through a different path. Furthermore, the parameters $C_{ij}, \bar{c}_{ij}, \alpha_{ij}, and \beta_{ij}$ also affect the suitability of an upgrade. Upgrading a link which already has a high capacity for vehicles has lesser effect compared to upgrading a low capacity link; it is sometimes more beneficial to upgrade and divert vehicles through a low capacity link than to upgrade a highly congested high capacity link.



(a) Original congestion. Darker links denote heavier congestion.

(b) Upgraded congestion. Darker links denote heavier congestion.



(c) Optimal network upgrade decisions. Upgraded links shown in black.

Figure 3: Congestion and upgrade decisions for the Sioux Falls transportation network.

2.3 Comparison with user equilibrium

The capital budget B was varied between 0 and 100 units for the Sioux Falls traffic network. Each link upgrade increased the effective capacity by 800 veh/h at a cost of 1 unit. For each budget level, the system optimal build decisions were fixed and the user equilibrium traffic assignment problem was computed on each modified network. Figure 4 shows the absolute system congestion as the budget varies.

When road users select routes to minimise their individual travel times, the network congestion is always greater than when guided by a central planner. This can be seen by the system optimal congestion being always less than the user equilibrium congestion. We see the effect of the non-convexity described in Section 1.3 appearing at a budget of 2700 units; as the budget increased, so too did the user equilibrium congestion.

It is worth reiterating that these user equilibrium congestion values are calculated with upgrades determined by system optimal user behaviour. If upgrade decisions are determined by a UENDP, we expect the congestion to lie below the values shown in white in Figure 4 but always greater than the lower bounds shown in black.

We see that upgrades selected based on system optimal user behaviour will, in general, improve the user equilibrium congestion.

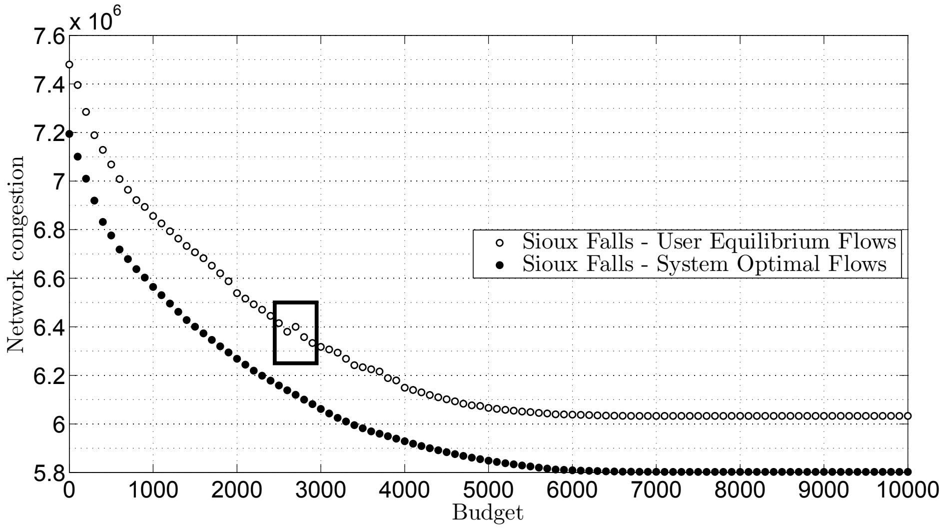
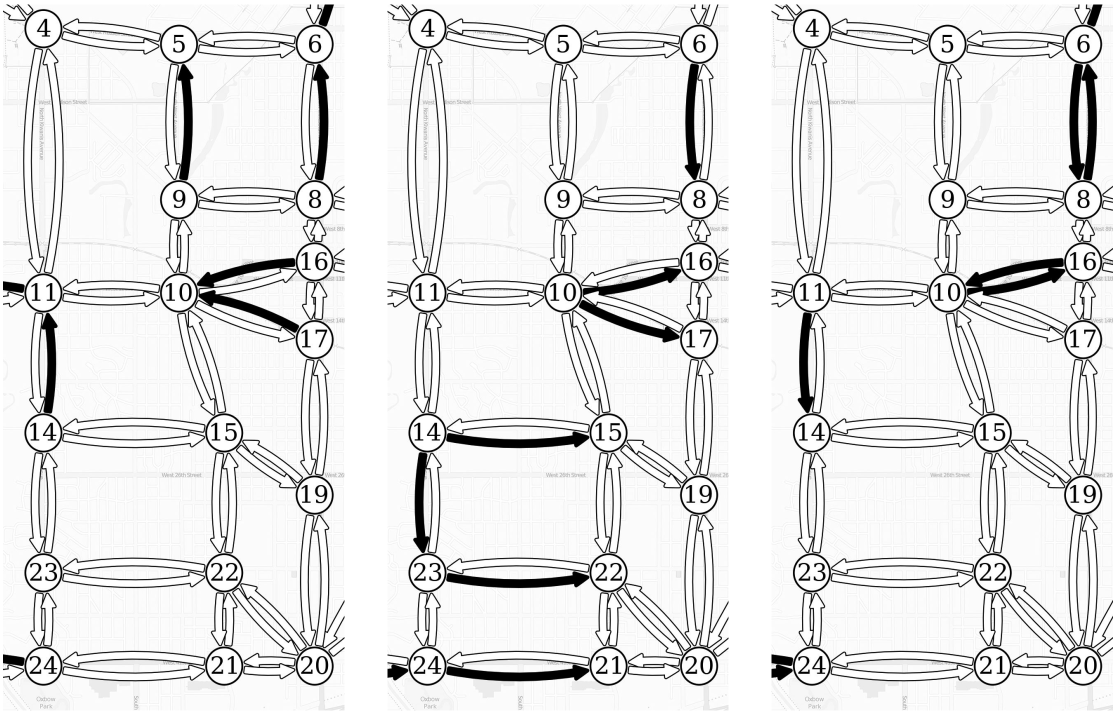


Figure 4: Network congestion using user equilibrium and system optimal flows, with upgrades made using the SONDP at varying budgets.

In order to reach the maximum reduction in congestion, tolls may be placed on links throughout the network to incentivise system optimal behaviour (Patriksson 1994). This specific details of how these optimal tolls can be implemented in practice is outside the scope of this project.

3 Extension to Multiple States

The SONDP determines the optimal links to upgrade in a traffic network under a particular loading scenario. However, loading conditions vary between different days of the week, and even within a single day. Typical traffic assignment models account for this by incorporating different *time of day matrices* (Davies, Valero, and Young 2009), which refer to the matrices storing point to point traffic demand, d_n^o. Each such matrix represents a distinct *state* of the network, which need to be considered in the optimisation model.



(a) Optimal upgrades for State 1.

(b) Optimal upgrades for State 2.

(c) Optimal upgrades for average of both states.

Figure 5: Optimising over different states provides different solutions.

Figure 5a and Figure 5b show optimal solutions to the SONDP when considering two states of the network individually. Figure 5c shows the optimal upgrade decisions when both network states are considered in unison. It can be seen that link (11,14) is not selected to be upgraded in any individual network state. However when both states are considered, its capacity is expanded. This demonstrates the necessity to optimise over multiple states at once, since some capacity expansion decisions may otherwise be missed.

3.1 Formulation

In order to extend the SONDP to incorporate multiple network states, the congestion for each individual state must be modelled. Let $\mathcal{S}$ denote the set of possible network states each with a weighting ω_s on the objective function, which can be interpreted as the importance of a particular state $s \in \mathcal{S}$. θ_{ij}^s now denotes the congestion on link $(i,j) \in \mathcal{A}$ in state s; $x_{ij}^{o,s}$ is the flow on link (i,j) for users originating from $o \in \mathcal{O}$ in state s; and f_{ij}^s is the total flow on each link (i,j) for each state s. The multiple state network design problem (MSNDP) can now be formulated as:

$$\min \quad \sum_{s\in\mathcal{S}} \sum_{(i,j)\in\mathcal{A}} \omega_s \theta_{ij}^s \tag{10}$$

$$\text{s.t.} \quad \sum_{(i,j)\in\mathcal{A}^*} b_{ij} z_{ij} \leq B \tag{11}$$

$$c_{ij} = C_{ij} + \bar{c} z_{ij} \qquad \forall (i,j) \in \mathcal{A} \tag{12}$$

$$\text{Traffic assignment constraints for state } s \text{ (5 -7)} \qquad \forall s \in \mathcal{S} \tag{13}$$

$$\text{Objective function Kelley's cutting planes for state } s \text{ (9)} \qquad \forall s \in \mathcal{S} \tag{14}$$

3.2 Benders decomposition

The matrix structure for this problem has a block structure which lends itself to decomposition methods. We randomly selected 30 links which formed $\mathcal{A}^*$, the set of upgradable links, and solved the network design problem over varying numbers of states for 10 replications.

Figure 6 compares the solve time between the large MIP and the Benders decomposition method. As the size of the network and number of states becomes large, the MIP quickly becomes intractable. The Benders decomposition appears to perform consistently well as the number of states increases.

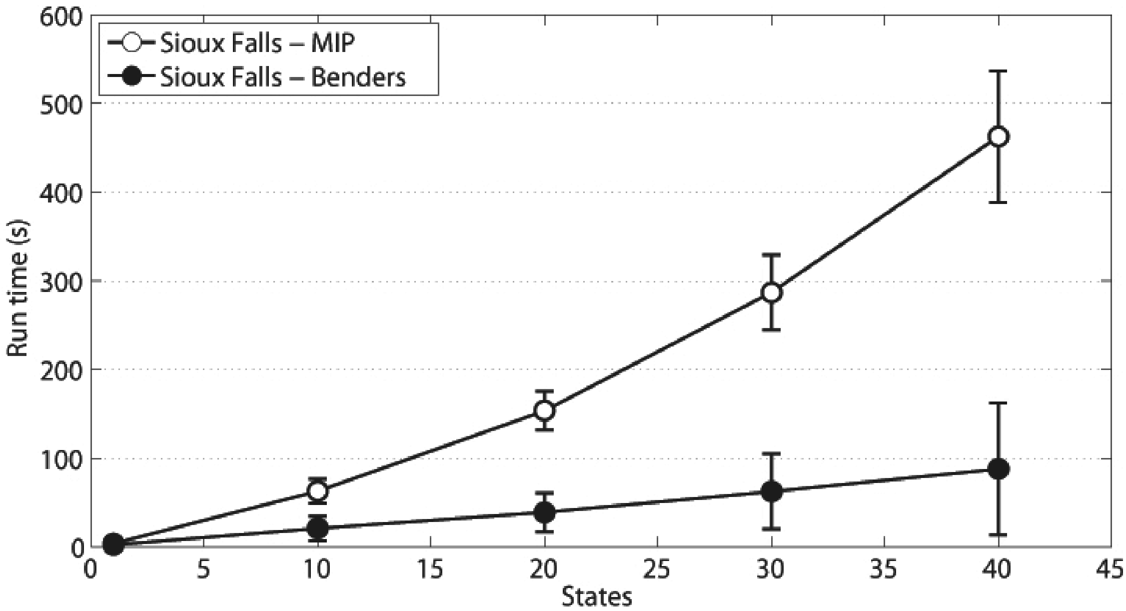


Figure 6: Solve time comparison between large MIP and Benders approach.

4 Conclusions

The network design problem was formulated assuming underlying system optimal flows and this model was tested on networks with varying budget levels.

- The solutions obtained from the system optimal network design represent the optimal upgrades to reduce the potential congestion in a traffic network.
- When user equilibrium flows are computed using these network upgrades, the congestion levels still showed improvements.
- Further improvements to the user equilibrium congestion can be made by incentivising users to behave closer to the system optimum through appropriate tolls.

The network design problem was extended to incorporate multiple network states. We implemented Benders' decomposition to solve the multi-state network design problem, and compared its performance with a large MIP formulation.

- Upgrade decisions when multiple states are considered can include choices which did not appear when optimising for a single state.

- When the number of decision variables is small, Benders' decomposition is able to reach the solution faster than the MIP.
- Benders' decomposition appears to scale better as the problem size increases; both in terms of the size of network as well as number of network states.
- As the dimensionality of the problem increases, the rate of convergence for Benders' decomposition deteriorates.

Acknowledgments

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References

Bar-Gera, Hillel. 2013, 3. Transportation Network Test Problems. `http://www.bgu.ac.il/~bargera/tntp/`.

Bureau of Public Roads. 1964. *Traffic assignment manual.*

Davies, J., J. Valero, and D. Young. 2009. "The Auckland Transport Models Project: Overview and use to date." *Australian Transport Research Forum 2009.*

Mathew, T., and S. Sharma. 2009. "Capacity Expansion Problem for Large Urban Transportation Networks." *Journal of Transportation Engineering* 135 (7): 406–415.

Patriksson, M. 1994. *The traffic assignment problem: models and methods.* Topics in transportation. VSP.

Sheffi, Y. 1984. *Urban Transportation Networks: Equilibrium Analysis With Mathematical Programming Methods.* Prentice-Hall.

Wardrop, J. 1952. "Some theoretical aspects of road traffic research." *Proceedings of the Institution of Civil Engineers, Part II* 1 (36): 352–362.