Transmuted exponentiated modified Weibull distribution with an application to bladder cancer data

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Abstract

This research investigates the potential usefulness of the transmuted exponentiated modified Weibull (TEMW) distribution for modelling bladder cancer data. This distribution, formed using the quadratic rank transmutation map technique, contains twenty three lifetime distributions as special cases. We obtain the analytic shapes of the density and hazard functions. Some structural properties of the transmuted exponentiated modified Weibull distribution are discussed, with special emphasis on its moments. We discuss estimation of the model parameters by the method of maximum likelihood and provide an application for bladder cancer data.

Key words: Exponentiated modified Weibull distribution; moments; maximum likelihood estimation.

1 Introduction

The statistics literature is filled with hundreds of lifetime distributions for describing and predicting real world phenomena. These distributions have been extensively used for modelling lifetime data in reliability engineering and biological studies. There are several methods to develop new family of lifetime distributions, such as the following the beta generated, Gamma generated, Kumaraswamy generated, exponentiated generated, T-X generated family of lifetime distributions and the recently introduced new family called transmuted generated family of continuous lifetime distributions which are more flexible for fitting in reliability and survival studies. Recently Elbatal (2011) introduced the exponentiated modified Weibull distribution for which the cumulative distribution function is given by

$$G(x) = \left\{1 - e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma}. \tag{1}$$

The probability density function corresponding to (1) is

$$g(x) = \gamma(\alpha + \eta\beta x^{\beta-1})e^{-(\alpha x+\eta x^{\beta})}\left\{1 - e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma-1}, \tag{2}$$

The literature presents many new distributions based on the transmuted technique which was originally proposed by Shaw and Buckley (2009). Aryal and Tsokos (2011) proposed the transmuted Weibull distribution and gave applications. Recently Khan and

King (2013) introduced the transmuted modified Weibull distribution and framed some of its properties within applications. Khan and King (2014) also proposed the transmuted inverse Weibull distribution and studied various structural properties with an application to survival data. More recently Khan, King and Hudson (2015) studied the various structural properties of the transmuted Weibull distribution and also proposed the log-transmuted Weibull distribution within the framework of covariates regression modelling. Elbatal and Aryal (2013) proposed the transmuted additive Weibull distribution with an application to reliability data. Ashour and Eltehiwy (2013) introduced the transmuted exponentiated modified Weibull distribution which seems to be superior over the baseline model for certain applications. Merovci (2013) proposed the transmuted Rayleigh distribution with an application to survival data. Yuzhu et al. (2014) introduced and studied the transmuted linear exponential distribution with application. This paper examines the moment estimation and statistical inference for modelling survival data. Recently Shaw and Buckley (2009) introduced the quadratic rank transmutation map (QRTM) technique for adding a new parameter to an existing distribution which satisfies the following relationships

$$F(x) = (1+\lambda)G(x) - \lambda G(x)^2, \qquad |\lambda| \le 1 \tag{3}$$

and

$$f(x) = g(x)[(1+\lambda) - 2\lambda G(x)], \tag{4}$$

where $G(x)$ is the cdf of the baseline distribution, $g(x)$ and $f(x)$ are the corresponding probability density functions (pdfs) associated with $G(x)$ and $F(x)$, respectively. In Section 2, we present the analytical shapes of the density and hazard functions. Moment estimation are addressed in Section 3. We implement the maximum likelihood estimation procedure to estimate the model parameters are discussed in Section 4. Bladder cancer data of Lee and Wang (2003) is analysed to illustrate the usefulness of the TEMW distribution and associated inferences made along with graphical comparisons of pp-plots which are displayed in Section 5. Concluding remarks are addressed in Section 6.

2 Transmuted exponentiated modified Weibull distribution

A random variable X has the transmuted exponentiated modified Weibull (TEMW) distribution with parameters $\alpha, \beta, \eta, \gamma > 0$ and $|\lambda| \le 1$, $x > 0$. The probability density function (pdf) and distribution function of the TEMW distribution are (see Ashour and Eltehiwy (2013))

$$f(x) = \gamma\left(\alpha + \eta\beta x^{\beta-1}\right)e^{-(\alpha x+\eta x^{\beta})}\left\{1 - e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma-1} \times \left[1 + \lambda - 2\lambda\left\{1 - e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma}\right], \tag{5}$$

and

$$F(x) = \left\{1 - e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma}\left\{1 + \lambda - \lambda\left\{1 - e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma}\right\}, \tag{6}$$

where α and η are the scale parameters, β and γ are the shape parameters and λ is the transmuted parameter.

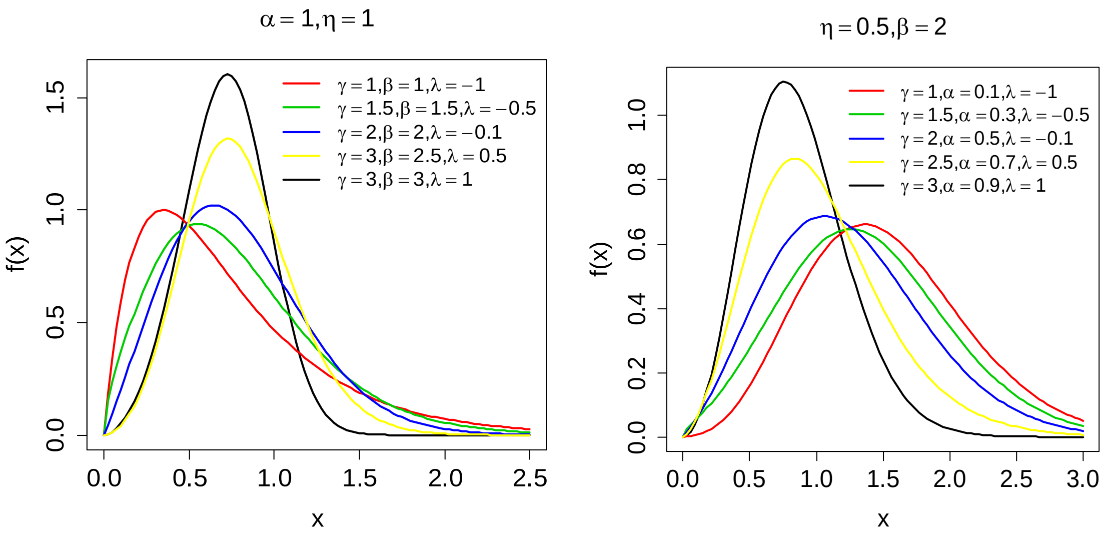


Figure 1. Plots of the TEMW pdf for some selected values of the parameters.

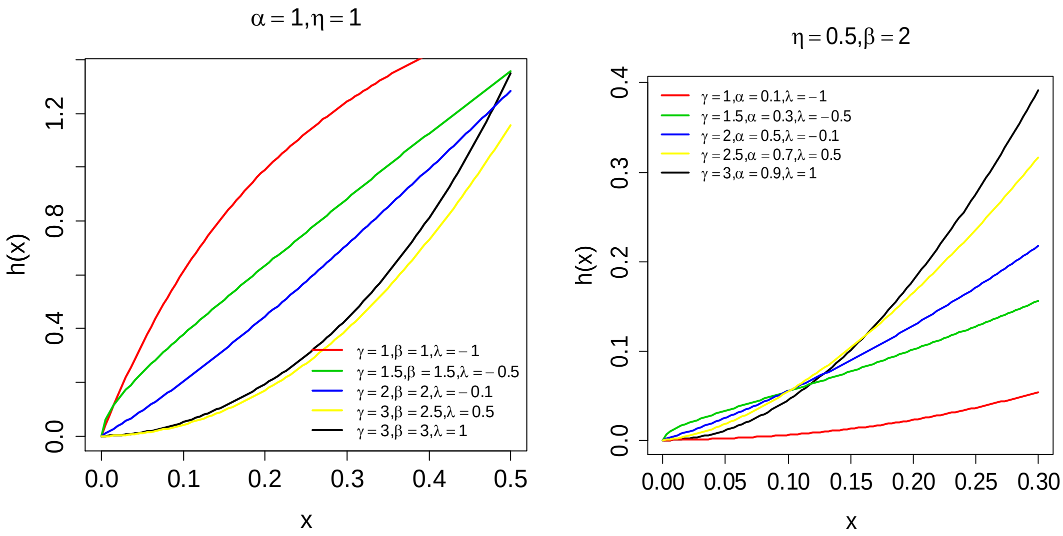


Figure 2. Plots of the TEMW hf for some selected values of the parameters.

If X is a random variable with density function (5), we write $X \sim TEMW(x;\alpha,\beta,\eta,\gamma,\lambda)$. The associated reliability and hazard rate functions which follow from (5) and (6) are

$$R(x) = 1 - \left\{1 - e^{-(\alpha x + \eta x^{\beta})}\right\}^{\gamma} \left\{1 + \lambda - \lambda \left\{1 - e^{-(\alpha x + \eta x^{\beta})}\right\}^{\gamma}\right\}, \qquad (7)$$

$$h(x) = \frac{\gamma\left(\alpha + \eta\beta x^{\beta-1}\right) e^{-(\alpha x + \eta x^{\beta})} \left\{1 - e^{-(\alpha x + \eta x^{\beta})}\right\}^{\gamma-1} \left\{1 + \lambda - 2\lambda \left\{1 - e^{-(\alpha x + \eta x^{\beta})}\right\}^{\gamma}\right\}}{1 - \left\{1 - e^{-(\alpha x + \eta x^{\beta})}\right\}^{\gamma} \left\{1 + \lambda - \lambda \left\{1 - e^{-(\alpha x + \eta x^{\beta})}\right\}^{\gamma}\right\}}, \qquad (8)$$

Plots of the probability density for the TEMW distribution for some chosen parameter values are given in Figure 1. The hazard functions of the TEMW distribution have different shapes dependent on the different choice of parameters as presented in Figure 2.

3 Moments

This section derives the moments of the TEMW distribution.

Theorem 1: If X has the $TEMW(x;\alpha,\beta,\eta,\gamma,\lambda)$ with $|\lambda| \leq 1$, then the k^{th} moment of X is

$$\acute{\mu}_k = (1+\lambda)\sum_{i,j=0}^{\infty}\binom{\gamma-1}{i}\frac{\eta^j\gamma(-1)^{i+j}}{j!\,(i+1)^{-j}}\left\{\frac{\alpha\Gamma(k+\beta j+1)}{\big(\alpha(i+1)\big)^{k+\beta j+1}}+\frac{\beta\eta\Gamma(k+\beta j+\beta)}{\big(\alpha(i+1)\big)^{k+\beta j+\beta}}\right\}$$
$$-2\lambda\sum_{i,j=0}^{\infty}\binom{2\gamma-1}{i}\frac{\eta^j\gamma(-1)^{i+j}}{j!\,(i+1)^{-j}}\left\{\frac{\alpha\Gamma(k+\beta j+1)}{\big(\alpha(i+1)\big)^{k+\beta j+1}}+\frac{\theta\eta\Gamma(k+\beta j+\beta)}{\big(\alpha(i+1)\big)^{k+\beta j+\beta}}\right\}.$$

Proof: By definition

$$\acute{\mu}_k = \int_0^{\infty} x^k\,\gamma\left(\alpha+\eta\beta x^{\beta-1}\right)e^{-(\alpha x+\eta x^{\beta})}\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma-1}$$
$$\left[1+\lambda-2\lambda\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma}\right]dx$$

then using (5) and (6), the above integral can be written as

$$\acute{\mu}_k = (1+\lambda)\int_0^{\infty} x^k\gamma\left(\alpha+\eta\beta x^{\beta-1}\right)e^{-(\alpha x+\eta x^{\beta})}\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma-1}dx$$
$$-2\lambda\int_0^{\infty} x^k\gamma\left(\alpha+\eta\beta x^{\beta-1}\right)e^{-(\alpha x+\eta x^{\beta})}\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{2\gamma-1}dx,$$

$$\acute{\mu}_k = (1+\lambda)\int_0^{\infty} x^k\gamma\alpha e^{-(\alpha x+\eta x^{\beta})}\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma-1}dx\,+$$
$$(1+\lambda)\int_0^{\infty} x^{k+\beta-1}\gamma\eta\beta e^{-(\alpha x+\eta x^{\beta})}\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma-1}dx$$
$$-2\lambda\int_0^{\infty} x^k\gamma\alpha e^{-(\alpha x+\eta x^{\beta})}\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{2\gamma-1}dx$$
$$-2\lambda\int_0^{\infty} x^{k+\beta-1}\gamma\eta\beta e^{-(\alpha x+\eta x^{\beta})}\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{2\gamma-1}dx,$$

From the binomial expansion the above equation reduces to

$$\acute{\mu}_k = (1+\lambda)\sum_{i=0}^{\infty}\binom{\gamma-1}{i}\gamma\alpha(-1)^i\int_0^{\infty}x^k e^{-(i+1)(\alpha x+\eta x^{\beta})}dx\,+$$
$$(1+\lambda)\sum_{i=0}^{\infty}\binom{\gamma-1}{i}\gamma\eta\beta(-1)^m\int_0^{\infty}x^{k+\beta+1}e^{-(i+1)(\alpha x+\eta x^{\beta})}dx$$

$$-2\lambda\sum_{i=0}^{\infty}\binom{2\gamma-1}{i}\gamma\alpha(-1)^{i}\int_{0}^{\infty}x^{k}e^{-(i+1)(\alpha x+\eta x^{\beta})}dx$$
$$-2\lambda\sum_{i=0}^{\infty}\binom{2\gamma-1}{i}\gamma\eta\beta(-1)^{i}\int_{0}^{\infty}x^{k+\beta+1}e^{-(i+1)(\alpha x+\eta x^{\beta})}dx,$$

Finally, we obtain

$$\acute{\mu}_k=(1+\lambda)\sum_{i,j=0}^{\infty}\binom{\gamma-1}{i}\frac{\eta^{j}\gamma(-1)^{i+j}}{j!\,(i+1)^{-j}}\left\{\frac{\alpha\Gamma(k+\beta j+1)}{(\alpha(i+1))^{k+\beta j+1}}+\frac{\beta\eta\Gamma(k+\beta j+\beta)}{(\alpha(i+1))^{k+\beta j+\beta}}\right\}$$
$$-2\lambda\sum_{i,j=0}^{\infty}\binom{2\gamma-1}{i}\frac{\eta^{j}\gamma(-1)^{i+j}}{j!\,(i+1)^{-j}}\left\{\frac{\alpha\Gamma(k+\beta j+1)}{(\alpha(i+1))^{k+\beta j+1}}+\frac{\theta\eta\Gamma(k+\beta j+\beta)}{(\alpha(i+1))^{k+\beta j+\beta}}\right\}. \quad (9)$$

The important feature and characterisations of the TEMW distribution can thus be studied using equation (9). The variance, coefficient of variation, skewness and kurtosis measures may be calculated from ordinary moments using well-known relationships. Table 1 lists the first eight moments of the TEMW distribution for some selected values of the parameters which can be calculated numerically through the R and SAS computational languages.

Table 1: Moments of the TEMW distribution for some selected values of parameters

$\acute{\mu}_k$	$\alpha=1,\beta=3,\eta=1,\gamma=2$			
	λ =-1	λ= -0.5	λ =0.5	λ =1
$\acute{\mu}_1$	0.9627	0.8691	0.6818	0.5882
$\acute{\mu}_2$	1.0050	0.8574	0.5623	0.4147
$\acute{\mu}_3$	1.1205	0.9225	0.5264	0.3284
$\acute{\mu}_4$	1.3213	1.0617	0.5427	0.2832
$\acute{\mu}_5$	1.6361	1.2924	0.6049	0.2613
$\acute{\mu}_6$	2.1162	1.6509	0.7202	0.2549
$\acute{\mu}_7$	2.8472	2.2006	0.9076	0.2611
$\acute{\mu}_8$	3.9709	3.0479	1.2019	0.2789
SD	0.2796	0.3195	0.3122	0.2621
CV	0.2905	0.3676	0.4578	0.4457
CS	0.1099	-0.0023	0.3334	0.2015
CK	2.9762	2.7717	2.8597	2.6047

4 Maximum Likelihood Estimation

Given a random samples $x_1, x_2,, x_n$ from the TEMW distribution with parameters $(\alpha,\beta,\eta,\gamma,\lambda)$, the log-likelihood function $\mathcal{L}$ = lnL of (5) is

$$\mathcal{L}=nln\gamma+\sum_{i=1}^{n}ln\left(\alpha+\eta\beta x^{\beta-1}\right)-\alpha\sum_{i=1}^{n}x+(\gamma-1)\sum_{i=1}^{n}ln\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}$$

$$-\eta\sum_{i=1}^{n}x^{\beta}+\sum_{i=1}^{n}ln\left\{1+\lambda-2\lambda\left\{1-e^{-(\alpha x+\eta x^{\beta})}\right\}^{\gamma}\right\}. \qquad (10)$$

By differentiating (10) with respect to α,β,η,γ and λ and then equating it to zero, we obtain the following estimating equations

$$\frac{\partial\mathcal{L}}{\partial\alpha}=\sum_{i=1}^{n}\frac{1}{(\alpha+\eta\beta x^{\beta-1})}-\sum_{i=1}^{n}x+\alpha(\gamma-1)\sum_{i=1}^{n}\frac{v_i}{1-v_i}$$

$$-2\lambda\alpha\gamma\sum_{i=1}^{n}\frac{(1-v_i)^{\gamma-1}v_i}{1+\lambda-2\lambda(1-v_i)^{\gamma}},$$

$$\frac{\partial\mathcal{L}}{\partial\beta}=\sum_{i=1}^{n}\frac{\eta x^{\beta-1}}{(\alpha+\eta\beta x^{\beta-1})}\left(\frac{\beta(\beta-1)}{x}+1\right)+\eta(\gamma-1)\sum_{i=1}^{n}\frac{x^{\beta}lnx\, v_i}{1-v_i}$$

$$-\eta\sum_{i=1}^{n}x^{\beta}\,lnx-\sum_{i=1}^{n}\frac{2\lambda\eta\gamma(1-v_i)^{\gamma-1}x^{\beta}lnxv_i}{1+\lambda-2\lambda(1-v_i)^{\gamma}},$$

$$\frac{\partial\mathcal{L}}{\partial\eta}=\sum_{i=1}^{n}\frac{\beta x^{\beta-1}}{(\alpha+\eta\beta x^{\beta-1})}-\sum_{i=1}^{n}x^{\beta}+(\gamma-1)\sum_{i=1}^{n}\frac{v_i x^{\beta}}{(1-v_i)}-\sum_{i=1}^{n}\frac{2\lambda\gamma(1-v_i)^{\gamma-1}x^{\beta}v_i}{1+\lambda-2\lambda(1-v_i)^{\gamma}},$$

$$\frac{\partial\mathcal{L}}{\partial\gamma}=\frac{n}{\gamma}+\sum_{i=1}^{n}(1-v_i)-\sum_{i=1}^{n}\frac{2\lambda(1-v_i)^{\gamma}ln(1-v_i)}{1+\lambda-2\lambda(1-v_i)^{\gamma}},$$

and

$$\frac{\partial\mathcal{L}}{\partial\lambda}=\sum_{i=1}^{n}\frac{1-2(1-v_i)^{\gamma}}{1+\lambda-2\lambda(1-v_i)^{\gamma}},$$

where $v_i=e^{-(\alpha x+\eta x^{\beta})}$. The maximum likelihood estimates (MLEs) can be obtained by solving these non-linear equations numerically in R and SAS, thus yielding the ML estimators $\hat{\alpha},\hat{\beta},\hat{\eta},\hat{\gamma}$ and $\hat{\lambda}$ of the TEMW distribution.

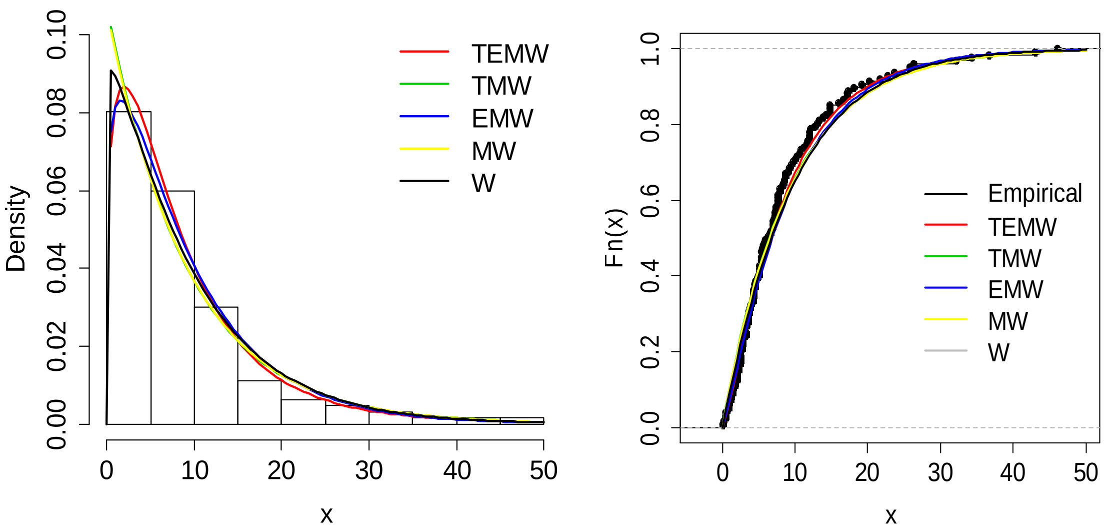
5 Bladder cancer application

This section provides the data analysis in order to assess the goodness-of-fit of the TEMW distribution in analysing the remission times (in months) of a random sample of 128 bladder cancer patients. The data have been obtained from Lee and Wang (2003). The Transmuted exponentiated modified Weibull, Transmuted modified Weibull, exponentiated modified Weibull, modified Weibull and Weibull distributions are now fitted to the cancer data of Lee and Wang (2003). The MLEs of the unknown parameters with their corresponding Akaike information criteria (AIC) for the fitted models are given in Table 2.

Table 2: MLEs of the Parameters for cancer patient's data and the AIC measure

Model	Parameter Estimates					AIC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\eta}$	$\hat{\gamma}$	$\hat{\lambda}$	
TEMW	0.0878	0.2443	0.0001	1.3026	0.6978	831.89
TMW	0.1068	0.3557	12E-8	-	0.0100	836.68
EMW	0.1212	0.1859	0.0001	1.2179	-	834.15
MW	0.1068	0.3484	11E-8	-	-	834.68
W	-	1.0478	0.0938	-	-	832.17

We have provided the parametric estimates of the transmuted exponentiated modified Weibull distribution and its requisite sub-models. Figure 3 gives the density functions with overlaid with a histogram and also the empirical fitted plots of the five distributions examined. Visual examination of the density functions show that the TEMW distribution provides better fit than the other four distributions (see Figure 3).



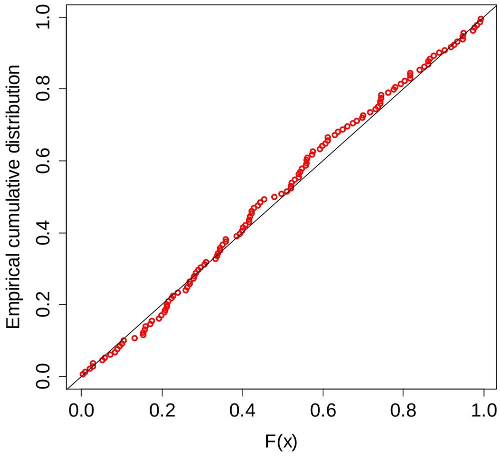
(a) Estimated pdf (b) Estimated cdf

Figure 3. Fitted models for remission times of cancer patient's data.

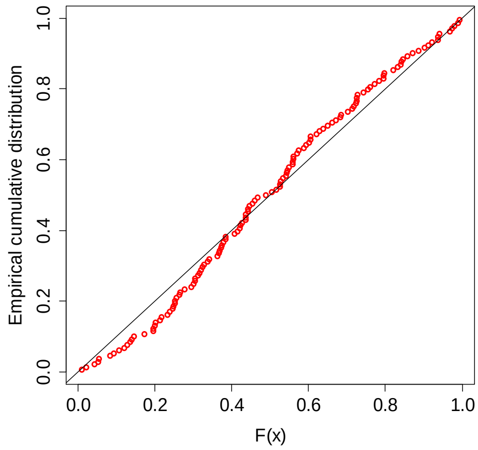
Table 3: The Log-likelihood, the values of likelihood ratio test (LRT) and P-values

Model	H_0	$-\ell(.;x)$	Λ	d.f	P-value
TMW	$\gamma = 1$	414.342	6.79	1	0.0092
EMW	$\lambda = 0$	413.077	4.26	1	0.0390
MW	$\gamma = 1, \lambda = 0$	414.342	6.79	2	0.0092
W	$\alpha = 0, \gamma = 1, \lambda = 0$	414.090	6.28	3	0.0122

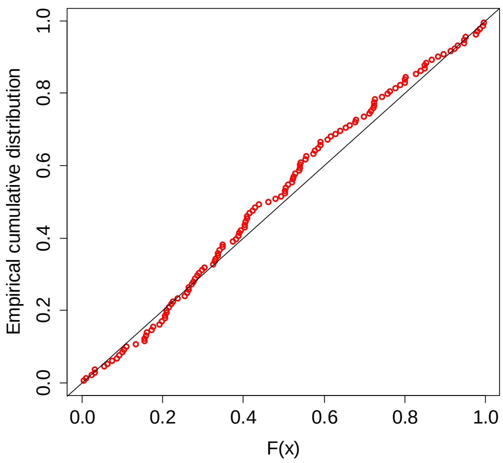
The criteria for comparisons of the TEMW distribution with the other four candidate lifetime distributions are shown in Table 2. The TEMW model under study has the smallest value of AIC which indicates that the TEMW distribution is the best model among the five fitted models. Hence the TEMW distribution exhibits a better relationship with the bladder cancer data and hence this distribution is potentially a good model for fitting lifetime data. Figure 3 (b) also shows that the TEMW distribution provides a better fit than the sub-models.



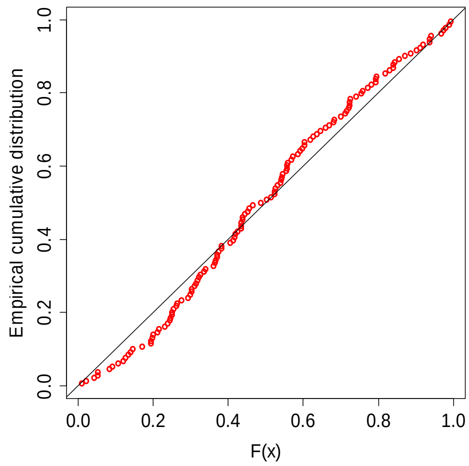
(a) TEMW



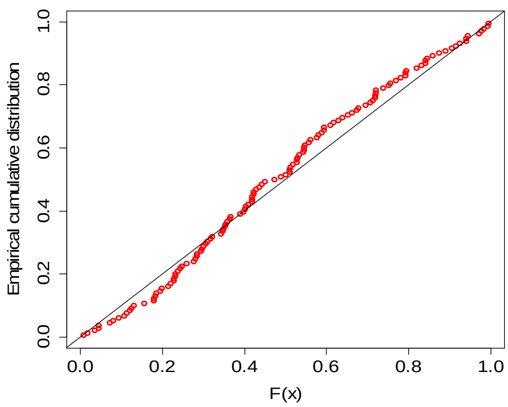
(b) TMW



(c) EMW



(d) MW



(e) W

Figure 4. P-P Plots of the TEMW, TMW, EMW, ME and W distributions for the bladder cancer data.

Further we calculate the maximum values of the log-likelihoods to obtain the likelihood Ratio test (LRT) for testing the hypothesis as displayed in Table 3. For example consider the LR statistics obtained for testing the hypothesis $H_0: \gamma = 1$ versus $H_1: H_0$ is not true, wherein we compare the TMW model with the TEMW model. Here the LR statistic for the TMW model is $\Lambda = 2[414.342 - 410.946] = 6.79$, p-value = 0.0092 yields a favourable indication for the TEMW distribution. As seen from Table 3 for the 5% level of significance ($\alpha = 0.05$) the null hypotheses are rejected in favour of the TEMW distribution. Furthermore, we apply the K-S test, the Cramér-von Mises and Anderson-Darling goodness of fit statistics in order to establish which model fits the best for the cancer remission data. Table 4 shows that the TEMW distribution has the smallest values of these goodness of fit statistics; therefore the TEMW model can be chosen as the best model. The pp-plots of all five fitted models are displayed in Figure 4. It is conclude that the TEMW distribution is the best model given its better fit as displayed on the pp-plot for the cancer data. We thus conclude that the TEMW model fits the cancer patient's remission data the best.

Table 4: The K-S test, Cramér-von Mises and Anderson-Darling goodness of-fit tests

Model	K-S test	$\mathcal{W}$	$\mathcal{A}$
TEMW	0.0521	0.0608	0.3663
TMW	0.0846	0.1193	0.7159
EMW	0.0725	0.1122	0.6741
MW	0.0846	0.1192	0.7159
W	0.0700	0.1314	0.7865

6 Concluding Remarks

In this paper we have studied the five parameter transmuted exponentiated modified Weibull distribution, which includes several distributions which are to date widely used in the statistics literature and is shown to be more flexible than the exponentiated modified Weibull model. We obtained the analytical shapes of both the density and hazard functions of the TEMW distribution. We formulated the moments of the TEMW distribution and obtained the moments values through numerical procedures in R. The flexibility and usefulness of the TEMW model is illustrated in an application to bladder cancer data and the parameters are estimated through maximum likelihood estimation. Based on the five goodness of fit measures, the results of the application indicate that the TEMW distribution performs better than the other four lifetime models.

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